NAVY, Caryn Linda
NONPARACOMPACTNESS IN PARA-LINDELÖF SPACES.

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NONPARACOMPACTNESS IN PARA-LINDELÖF SPACES

A thesis submitted to the Graduate School of the University of Wisconsin-Madison in partial fulfillment of the requirements for the degree of Doctor of Philosophy

by

CARYN LINDA NAVY

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Approved by Thesis Reading Committee:

Mary Ellen Rudin
Major Professor

June 24, 1981
Date of Examination

H. Jerome Keisler

James W. Cannon

Robert M. Bock
Dean, Graduate School

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Dedicated to my housemates — David, Jesse, Nevin, and Whiskey — who have put up with my work habits and encouraged me through all the phases of this project.

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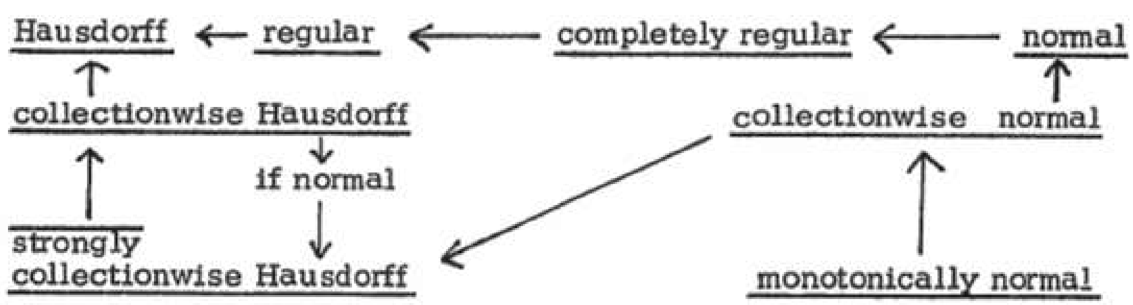
I would also like to thank Diane Reppert for her excellent job of typing.

Introduction.

A topological space is said to be para-Lindelöf if every open cover of it has a locally countable open refinement. The subject of locally finite collections of open sets and paracompact spaces has been fairly well developed for some time. However, the understanding of their weaker counterparts, locally countable collections of open sets and para-Lindelöf spaces, is relatively sketchy. The developments in this area upon which this paper builds, due mainly to Fleissner and Reed and also extended by Burke, are rather recent. The main question answered in this paper, asked by Fedorcuk [10], Tall [14], Fleissner [1], Reed [2], Burke [5], and Nyikos [6], is: Must every para-Lindelöf T_3-space be paracompact? The answer is no (not even completely regular or metacompact), as we will demonstrate with a variety of examples.

Para-Lindelöf spaces must be assumed to be T_3 to provide an interesting theory, for there is a space with a countable base which is Hausdorff but not regular. On the other hand, paracompactness richly endows T_2-spaces with all of the below separation properties, connected in T_1-spaces by the in the indicated implications:



As shown by Fleissner and Reed [2], para-Lindelöf T_3-spaces are collectionwise Hausdorff, in fact strongly so, and are thus of interest to those studying the normal Moore space problem. Included among the spaces we will construct are para-Lindelöf T_1-spaces which are respectively regular but not completely regular, completely regular but not normal, and normal but not collectionwise normal. We will also consider some of the other covering properties enjoyed by paracompact T_2-spaces, such as countable paracompactness, metacompactness, and screenability.

In Chapter 2 there is a brief review of some basic facts concerning para-Lindelöf spaces. Chapter 3 discusses the techniques used in the search for a para-Lindelöf, nonparacompact T_3-space, due mainly to Fleissner [1]. The paper's main examples of such spaces are presented in Chapters 4 and 5. In Chapter 6, the information gained from these examples is amplified, and some related unanswered research questions are asked.

Section 2.1. Conventions.

Except where otherwise specified, the letters i, j, m, and n will be reserved for nonnegative integers. The symbol $\subset$ will denote inclusion, not necessarily proper inclusion. For the meaning "belonging to," the symbol $\in$ and the word "in" will be used interchangeably. The symbol $\restriction$ will denote restriction. Ordered pairs will be notated with ordinary parentheses.

The symbol $\frown$ will be used with the meaning "tied to" as follows. Let g be a function with domain an ordinal γ and with range a set of ordinals. For an ordinal α, $g^\frown\alpha$ will denote the new function $g' = g \cup \{(\gamma, \alpha)\}$. That is, g' has domain $\gamma + 1$, $g' \restriction \gamma = g$, and $g'(\gamma) = \alpha$.

The symbol $\square$ will mark the end of each definition, example, or proof. Possibly unfamiliar topological properties appearing in the text will be defined in the glossary. Some of the less familiar ones will be defined in the text as well.

The set theory assumed in this paper is ZFC. Results requiring additional axioms of set theory will be so indicated. In particular, some of the examples use $MA(\omega_1)$, part of $(MA + \neg CH)$ (where MA stands for Martin's axiom and CH for the continuum hypothesis). A topological equivalent of $MA(\omega_1)$, reminiscent of the Baire category theorem, is: In a compact, c.c.c. Hausdorff space, the intersection of any collection of ω_1-many dense open sets is nonempty. Note that a topological space is (has the) c.c.c. (countable chain condition) provided there is no collection of uncountably many mutually disjoint open subsets of the space.

Section 2.2. Para-Lindelöf Spaces: Basic Facts

2.2.1: Definitions.

(a) A topological space X is para-Lindelöf if every open cover of X has a locally countable open refinement. A topological space X is σ-para-Lindelöf if every open cover of X has an open refinement which is σ-locally countable (i.e., the union of countably many locally countable collections).

(b) A collection $\mathscr{S}$ of subsets of a topological space X is discrete if every point of X has a neighborhood which meets (i.e., has nonempty intersection with) at most one member of $\mathscr{S}$. A collection of points of X is discrete if the associated collection of singletons is discrete.

(c) A T_1-space X is collectionwise Hausdorff (respectively, strongly collectionwise Hausdorff) if for every discrete collection $\{x_\alpha \mid \alpha \in A\}$ of points of X, there is a mutually disjoint (respectively, discrete) collection $\{U_\alpha \mid \alpha \in A\}$ of open sets with $x_\alpha \in U_\alpha$ for each α in A. □

The many separation properties which paracompactness implies for Hausdorff spaces provide a good means for comparing paracompact and para-Lindelof T_3-spaces. A more complete treatment of the positive separation results for para-Lindelof T_3-spaces is given by Fleissner and Reed in [2]. One of their important results of this kind is that these spaces are strongly collectionwise Hausdorff. A proof of this, which indicates the flavor of arguments using local countability, is included here.

<u>2.2.2: Lemma.</u> Let H and K be subsets of a topological space X. For any h in H and k in K, say $h * k$ and $k * h$ provided $h \neq k$. Let $S(x)$ be an open neighborhood of x for each x in $H \cup K$. Suppose that for each x in $H \cup K$, there is no element x' of $H \cup K$ such that $x' * x$ and $x' \in \overline{S(x)}$. Suppose also that for each x in $H \cup K$, there are only countably many points x' in $H \cup K$ with $x' * x$ for which $S(x) \cap S(x') \neq \emptyset$. Then each $S(x)$ can be refined to an open neighborhood $R(x)$ of x so that the collection $\{R(x) | x \in H \cup K\}$ satisfies the following. For each $R(x)$, there is no set $R(x')$ such that $x * x'$ and $R(x) \cap R(x') \neq \emptyset$.

<u>Proof.</u> Let $\sim$ be the equivalence relation on $H \cup K$ generated by the rule $x \sim x'$ if $x * x'$ and $S(x) \cap S(x') \neq \emptyset$. Let $E = (H \cup K)/\sim$, the set of $\sim$-equivalence classes on $H \cup K$. Notice that each such equivalence class is countable. Enumerate the members of each such class e as $x_{e,0}, x_{e,1}, x_{e,2}, \cdots$ (finitely many or ω-many as needed). For each $x_{e,n}$, let $R(x_{e,n}) = S(x_{e,n}) - \bigcup\{\overline{S(x_{e,j})} | j < n$ and $x_{e,j} * x_{e,n}\}$. $\square$

<u>2.2.3: Theorem.</u> Let X be a para-Lindelof T_3-space. Then:

(a) X is collectionwise Hausdorff.

(b) X is strongly collectionwise Hausdorff.

<u>Proof.</u> (a): Let $X_0 = \{x_\alpha | \alpha \in A\}$ be a discrete collection of points of X. By regularity of X, for each α in A let

U_α be an open neighborhood of x_α such that $\overline{U}_\alpha \cap X_0 = \{x_\alpha\}$. Since X is para-Lindelöf, the open cover $\{U_\alpha \mid \alpha \in A\} \cup \{X - X_0\}$ has a locally countable open refinement $\mathcal{V}$. For each α in A let V_α be a neighborhood of x_α belonging to $\mathcal{V}$. Since the collection $\mathcal{V}_0 = \{V_\alpha \mid \alpha \in A\}$ is locally countable, for each α in A let V'_α be an open subset of V_α which witnesses this at x_α; that is, $x_\alpha \in V'_\alpha$ and V'_α meets only countably many members of $\mathcal{V}_0$. Then the collection of open sets $\mathcal{V}'_0 = \{V'_\alpha \mid \alpha \in A\}$ is star-countable; that is, each set V'_α avoids all but countably many members of $\mathcal{V}'_0$.

Now we can apply Lemma 2.2.2 with $H = X_0$, $K = X_0$, and $S(x_\alpha) = V'_\alpha$ for each α in A; let the sets $R(x_\alpha) \subset S(x_\alpha)$ satisfy the lemma's conclusion. Let $W_\alpha = R(x_\alpha)$ for each α in A. Then $\mathcal{W}_0 = \{W_\alpha \mid \alpha \in A\}$ is a collection of mutually disjoint open sets with $x_\alpha \in W_\alpha$ for each α in A. Therefore X is collectionwise Hausdorff.

(b): Continue from (a). Let $\mathcal{W}'$ be an open cover which testifies that $\mathcal{W}_0$ is locally countable (for example, any open cover witnessing the local countability of $\mathcal{V}_0$). Now let $\mathcal{Z}$ be a locally countable open refinement of $\mathcal{W}'$. For each α in A let Z'_α be an open set witnessing the local countability of $\mathcal{Z}$ at x_α; using the regularity of X, assume without loss of generality that $\overline{Z'_\alpha} \subset W_\alpha$.

To prepare for an application of Lemma 2.2.2, let $H = X_0$ and $K = X - \bigcup \mathcal{W}_0$. For each α in A let $S(x_\alpha) = Z'_\alpha$. Finally, for each x in K let $S(x)$ be a member of $\mathcal{Z}$ with

$x \in S(x)$. Since these satisfy the hypotheses of the lemma, let the sets $R(x)$ for x in $H \cup K$ be as in its conclusion. Now let $G_\alpha = R(x_\alpha)$ for each α in A and $\mathcal{G}_0 = \{\mathcal{G}_\alpha \mid \alpha \in A\}$. If $x \in K$, then $R(x)$ meets no member of $\mathcal{G}_0$. Otherwise, for one α in A, x lies in W_α, which meets just one member of $\mathcal{G}_0$ — namely G_α. Thus $\mathcal{G}_0$ is discrete, and X is therefore strongly collectionwise Hausdorff. □

A number of results about paracompactness suggest similar consideration of para-Lindelöfness. For example, by a familiar theorem of E. Michael, every σ-paracompact T_3-space is actually paracompact. The comparable situation regarding para-Lindelöf spaces is somewhat different.

<u>2.2.4: Example.</u> The Moore space constructed by William Fleissner in [1] is σ-para-Lindelöf but not para-Lindelöf. □

<u>2.2.5: Theorem</u> (Fleissner and Reed [2]). If a T_3-space is countably paracompact and σ-para-Lindelöf, then it is para-Lindelöf. □

Some standard theorems about preservation of paracompactness translate to similar theorems about preservation of para-Lindelöfness.

<u>2.2.6: Theorems</u>.

(a) A closed subspace of a para-Lindelöf space is para-Lindelöf. If the space is T_3 and countably paracompact, we can replace closed by F_σ (using the just-mentioned result of Fleissner and Reed).

(b) The product of a para-Lindelöf space with a compact space is para-Lindelöf.

Likewise, some standard examples showing nonpreservation of paracompactness also reveal nonpreservation of a para-Lindelöfness.

2.2.7: Examples.

(a) An open subspace of a compact Hausdorff space need not be para-Lindelöf. An example of this is ω_1 as a subspace of the ordinal space $\omega_1 + 1$. In fact, ω_1 is not even meta-Lindelöf.

(b) The product of a Lindelöf space with a Lindelöf metric space need not be para-Lindelöf. An example of this is Michael's Product Topology [11, p. 105]. □

Another property using local countability, that of having a σ-locally countable base, is closely linked with metrizability. V. Fedorcuk [10] proved that every paracompact T_2-space having a σ-locally countable base is actually metrizable. Like the property of being para-Lindelöf, this property is not understood as well as its local finiteness counterpart (i.e., having a σ-locally finite base, which for T_3-spaces is equivalent to being metrizable). Finally, neither of these two local countability properties implies the toher.

2.2.8: Examples.

(a) The Moore space constructed by Fleissner in [1] has a σ-locally countable base but is not para-Lindelöf.

(b) By Fedorcuk's theorem, any paracompact but nonmetrizable T_2-space is para-Lindelöf but does not have a σ-locally countable base.

An example is the Sorgenfrey line. □

However, as Fleissner and Reed point out in [2], it is easy to see the following.

<u>2.2.9: Theorem</u>. Every para-Lindelöf Moore space has a σ-locally countable base. □

Section 3.1. The Approach.

The existence of a T_3, para-Lindelöf, nonparacompact space is equivalent to the existence of such a space which is not normal. This was pointed out by Van Mill as a consequence of the following familiar theorem of Tamano.

3.1.1: Theorem. For any Tychonoff space X, X is paracompact iff $X \times \beta X$ is normal. $\square$

To obtain Van Mill's result, suppose X is T_3 and para-Lindelöf but not paracompact. By Tamano's theorem, if X is normal (and hence Tychonoff), then $X \times \beta X$ is again T_3 and para-Lindelöf but not normal. In light of this result, attempting to construct a T_3, para-Lindelöf, nonnormal space seems like a natural approach to the main problem.

Fleissner points out in [1] what a T_3, para-Lindelöf, nonnormal space X must look like. "To start with, X must contain two disjoint closed sets H and K which cannot be separated. Because X is regular, there is a cover $\mathcal{U}_0$ of H by open sets whose closures miss K, and similarly a cover $\mathcal{V}_0$ of K by open sets whose closures miss H." Since X is para-Lindelöf and $\mathcal{U}_0 \cup \mathcal{V}_0 \cup \{X - (H \cup K)\}$ is an open cover of X, $\mathcal{U}_0$ and $\mathcal{V}_0$ have open refinements $\mathcal{U}_1$ and $\mathcal{V}_1$, respectively, which make $\mathcal{U}_0 \cup \mathcal{V}_1 \cup \{X - (H \cup K)\}$ a locally countable cover of X.

Although $\mathcal{U}_1 \cup \mathcal{V}_1$ is locally countable, it cannot be star-countable. That is, it cannot witness its own local countability at the points of $H \cup K$. For the techniques used to prove Theorem 2.2.3 show that if every member of $\mathcal{U}_1$ meets only countably many members of $\mathcal{V}_1$, and simultaneously every member of $\mathcal{V}_1$ meets only countably many members of $\mathcal{U}_1$, then H and K can be separated by disjoint open sets.

As Fleissner continues, "Because $(\mathcal{U}_1 \cup \mathcal{V}_1)$ is locally countable, there is a cover of X by open sets each of which meets only countably many elements of $(\mathcal{U}_1 \cup \mathcal{V}_1)$. This cover can be refined so that it refines $\mathcal{U}_1 \cup \mathcal{V}_1 \cup \{X - (H \cup K)\}$ and is itself locally countable. Repeating this process, there are," for every $n \geq 1$, "locally countable open covers $\mathcal{U}_{n+1}$ of H and $\mathcal{V}_{n+1}$ of K which refine $\mathcal{U}_n$ and $\mathcal{V}_n$, respectively and witness the local countability of $\mathcal{U}_n \cup \mathcal{V}_n$" at the points of $H \cup K$.

This picture of a T_3, para-Lindelöf, nonnormal space, obtained by considering conditions true in every such space, is very general. To it Fleissner adds detail by making some assumptions which are helpful in working toward a specific example.

"We assume that the points of $X - (H \cup K)$ are isolated. We assume that," for each $n \geq 1$, "$\mathcal{U}_n$ and $\mathcal{V}_n$ are families of ω_1-many disjoint clopen sets. We assume that each $U \in \mathcal{U}_{n+1}$ meets ω-many V's $\in \mathcal{V}_n$ and ω_1-many V's $\in \mathcal{V}_{n+1}$; similarly for $V \in \mathcal{V}_{n+1}$. Finally, we assume that

$$\bigcup_{n\in\omega} (\mathcal{U}_n \cup \mathcal{V}_n) \cup \{\{x\} : x \in X - (H \cup K)\}$$

is a base for X." Also, a consequence of these assumptions of of Fleissner's is the fact that each $U \in \mathcal{U}_n$ contains ω_1-many members of $\mathcal{U}_{n+1}$; similarly for each $V \in \mathcal{V}_n$.

Fleissner then points out that, with these assumptions, we can be more specific in describing X. Reworking his presentation slightly, we enumerate the ω_1-many disjoint members of $\mathcal{U}_1$ as $\{U_1(\alpha_0) | \alpha_0 < \omega_1\}$; similarly we enumerate $\mathcal{V}_1$ as $\{V_1(\beta_0) | \beta_0 < \omega_1\}$. Then, for every $U_1(\alpha_0) \in \mathcal{U}_1$, we enumerate the ω_1-many disjoint members of $\mathcal{U}_2$ contained in it as $\{U_2(\alpha_0, \alpha_1) | \alpha_1 < \omega_1\}$; we enumerate $\mathcal{V}_2$ similarly. This enumeration scheme reflects the fact that if $U_2(\alpha_0, \alpha_1)$ meets $V_2(\beta_0, \beta_1)$, then $U_1(\alpha_0)$ meets $V_1(\beta_0)$. Continuing in this manner, for every $n \geq 1$, we enumerate $\mathcal{U}_{n+1}$ as $\{U_{n+1}(\alpha_0, \cdots, \alpha_n) | \alpha_0, \cdots, \alpha_n < \omega_1\}$ where $U_{n+1}(\alpha_0, \cdots, \alpha_{n-1}, \alpha_n)$ is contained in $U_n(\alpha_0, \cdots, \alpha_{n-1})$; we enumerate $\mathcal{V}_{n+1}$ similarly.

Each $h \in H$ lies in exactly one $U \in \mathcal{U}_{n+1}$ for each $n \in \omega$. Thus each $h \in H$ determines a function $\hat{h} : \omega \to \omega_1$ by the requirement that $\hat{h}(n) = \alpha_n$ where $h \in U_{n+1}(\alpha_0, \cdots, \alpha_n)$. Likewise each $k \in K$ lies in exactly one $V \in \mathcal{V}_{n+1}$ for each $n \in \omega$ and determines a function $\hat{k} : \omega \to \omega_1$. In addition, for any $n \in \omega$ and $U = U_{n+1}(\alpha_0, \cdots, \alpha_n) \in \mathcal{U}_{n+1}$, U determines a function $\sigma_U : n+1 \to \omega_1$ defined by $\sigma_U(i) = \alpha_i$ for $0 \leq i \leq n$. Likewise each $V \in \mathcal{V}_{n+1}$ determines a function $\tau_V : n+1 \to \omega_1$. Notice that $h \in U$ iff σ_U restricts $\hat{h}$; similarly $k \in V$ iff τ_V restricts $\hat{k}$.

To simplify the combinatorics needed to describe X, we now recursively "spread out" the enumeration scheme for $\mathcal{U}_{n+1}$ and $\mathcal{V}_{n+1}$ for every $n \geq 1$. More precisely, we label each $U \in \mathcal{U}_2$ contained in $U_1(\alpha_0)$ as $U_2(\alpha_0, \alpha_1)$ for some α_1 satisfying $\alpha_1 > \alpha_0$ and, in addition, $\alpha_1 > \beta_0$ for every $V_1(\beta_0)$ which meets U. This is possible since U meets only countably many members of $\mathcal{V}_1$. We enumerate $\mathcal{V}_2$ similarly. In general, we label each $U \in \mathcal{U}_{n+1}$ contained in $U_n(\alpha_0, \cdots, \alpha_{n-1})$ as $U_{n+1}(\alpha_0, \cdots, \alpha_{n-1}, \alpha_n)$ where $\alpha_n > \alpha_{n-1}$ and $\alpha_n > \beta_{n-1}$ for every $V_n(\beta_0, \cdots, \beta_{n-1})$ meeting U; we enumerate $\mathcal{V}_{n+1}$ similarly.

With this new enumeration scheme, each $h \in H$, $k \in K$ represents an increasing function $\hat{h}, \hat{k} : \omega \to \omega_1$. Also, for each $n \geq 1$, each $U \in \mathcal{U}_n$, $V \in \mathcal{V}_n$ represents an increasing function $\sigma_U, \tau_V : n \to \omega_1$. Furthermore, if $U = U_n(\alpha_0, \cdots, \alpha_{n-1})$ meets $V = V_n(\beta_0, \cdots, \beta_{n-1})$, then the functions σ_U and τ_V "interlace". That is, $\alpha_i < \beta_{i+1}$ and $\beta_i < \alpha_{i+1}$ for $0 \leq i \leq n-2$.

These considerations, together with some sort of device to make the members of each $\mathcal{U}_n$ and $\mathcal{V}_n$ closed, lead fairly naturally to the space defined by Fleissner in [1]. In this space the points of $X-(H \cup K)$ are the ordered pairs (σ_U, τ_V) of interlacing functions arising from intersecting basic open sets $U \in \mathcal{U}_n$ and $V \in \mathcal{V}_n$ for some $n \geq 1$. However, the space in [1] is not para-Lindelöf. Problems occur in trying to find a locally countable open refinement for a cover which uses members of $\mathcal{U}_n \cup \mathcal{V}_n$ for infinitely many integers n.

To avoid this problem, another technique is used to thin down the basic open sets in each $\mathcal{U}_n$ and $\mathcal{V}_n$ and thus limit the occurrence of nonempty intersections of these sets. The basic open sets in each $\mathcal{U}_n$ and $\mathcal{V}_n$ are made thinner by associating them with open subsets of a reference space which is T_4 but not collectionwise Hausdorff.

Section 3.2. The Combinatorics.

The combinatorial tool developed by Fleissner in trying to build a T_3, para-Lindelöf, nonnormal space is that of full sets. In [1] he introduces full sets and develops their basic properties. Full sets are very important in proving that my examples are not paracompact — some not normal, some not collectionwise normal. Stationarily full sets were treated by Charles Mills [4] and Fleissner [1].

<u>3.2.1: Notational Definition</u>. Let $F = \{f : \omega \to \omega_1 \mid f \text{ is increasing}\}$. Given $n \in \omega$, let $P_n = \{\rho : n \to \omega_1 \mid \rho \text{ is increasing}\}$. Let $P = \bigcup_{n \geq 1} P_n$. □

<u>3.2.2: Definition.</u> Given $n \geq 1$, σ and τ in P_n are said to <u>interlace</u> if $\sigma(i) < \tau(i+1)$ and $\tau(i) < \sigma(i+1)$ for $0 \leq i \leq n-2$. □

<u>3.2.3: Notational Definition.</u> Given ρ in ${}^{<\omega}\omega_1$ and a subset A of ${}^{<\omega}\omega_1$, let $[\rho : A] = \{\sigma \in A \mid \rho \subset \sigma\}$, the set of extensions of ρ occurring in A. □

<u>3.2.4: Notational Definition</u>. Given $n \in \omega$, let A be a subset of ${}^{n}\omega_1$. Given $m \leq n$, let $\mathrm{pr}^m A = \{\sigma \upharpoonright m \mid \sigma \in A\}$, the set of those predecessors (initial restrictions) of members of A which have domain m. As in [1], let $\mathrm{pr} A = \bigcup_{m \leq n} \mathrm{pr}^m A$, the set of all predecessors of members of A. □

3.2.5: Definition. Let ρ be in ${}^{m}\omega_1$ for some $m \in \omega$. The set $\{\rho\}$ is called 0-full over ρ. A subset A of ${}^{m+1}\omega_1$ is called 1-full over ρ if $\sigma \restriction m = \rho$ for every σ in A and A is uncountable. Given $n \geq 1$, a subset A of ${}^{m+n+1}\omega_1$ is recursively defined to be (n+1)-full over ρ if $pr^{m+n}A$ is n-full over ρ and $[\sigma : A]$ is 1-full over σ (i.e., uncountable) for every σ in $pr^{m+n}A$. In other words, a subset A of ${}^{m+n}\omega_1$ is n-full over ρ iff every member of A extends ρ and every member of $pr^{m+i}A$ has uncountably many extensions in $pr^{m+i+1}A$ for $0 \leq i \leq n-1$. Finally, given $n \in \omega$, a subset A of ${}^{n}\omega_1$ is called n-full if it is n-full over the empty function $\emptyset$, the only member of ${}^{0}\omega_1$. □

3.2.5': Definition. Let ρ be in ${}^{m}\omega_1$ for some $m \in \omega$. The set $\{\rho\}$ is called stationarily 0-full over ρ. A subset A of ${}^{m+1}\omega_1$ is called stationarily 1-full over ρ if $\sigma \restriction m = \rho$ for every σ in A and $\{\alpha \in \omega_1 | \rho^\frown \alpha \in A\}$ is (not only uncountable but also) stationary in ω_1. Given $n \geq 1$, a subset A of ${}^{m+n+1}\omega_1$ is recursively defined to be stationarily (n+1)-full over ρ if $pr^{m+n}A$ is stationarily n-full over ρ and $[\sigma : A]$ is stationarily 1-full over σ for every σ in $pr^{m+n}A$. Finally, given $n \in \omega$, a subset A of ${}^{n}\omega_1$ is called stationarily n-full if it is stationarily n-full over $\emptyset$. □

<u>3.2.6: Definition</u>. Let η be in ${}^{m}\omega_1$ for some $m \in \omega$. Given $n \geq 1$ and a subset A of ${}^{m+n}\omega_1$, the family $(r_\rho : \rho \in A)$ is called an <u>n-full Δ-system over η</u> if A is n-full over η and, for every ρ and θ in A, the intersection of r_ρ and r_θ depends only on the maximal common predecessor of ρ and θ. More precisely, the family $(r_\rho : \rho \in A)$ can be expanded to a (unique) <u>root family</u> $(r_\sigma : \sigma \in \mathrm{pr}A,\ \mathrm{dom}(\sigma) \geq m)$ such that $r_\rho \cap r_\theta = r_\sigma$ when ρ and θ in A have σ as their maximal common predecessor. Notice that if $(r_\rho : \rho \in A)$ is a 1-full Δ-system over η with root family $(r_\sigma : \sigma \in \mathrm{pr}A,\ \mathrm{dom}(\sigma) \geq m)$, the collection $\{r_\rho \mid \rho \in A\}$ is a Δ-system in the usual sense with root r_η. Finally, the family $(r_\rho : \rho \in A)$ is called an <u>n-full Δ-system</u> if it is an n-full Δ-system over $\emptyset$. □

<u>3.2.7: Lemma</u>. Given $m \in \omega$, let ρ be in ${}^{m}\omega_1$. Given $n, j \in \omega$, let A be a subset of ${}^{m+n+j}\omega_1$. Then A is (stationarily) $(n+j)$-full over ρ iff $A = \bigcup_{\sigma \in D} C_\sigma$, where D is (stationarily) n-full over ρ and C_σ is (stationarily) j-full over σ for every σ in D.

<u>Proof.</u> Let $D = \mathrm{pr}^{m+n}A$, and let $C_\sigma = [\sigma : A]$ for every σ in D. □

<u>3.2.8: Lemma</u>. Given $n \in \omega$, let A be (stationarily) n-full. Then $A \cap P_n$, the set of increasing

members of A, is also (stationarily) n-full. □

3.2.9: Lemma. Let A be an n-full subset of P_n for some $n \geq 1$. Then there exist mutually interlacing members of A, σ_i for every $i \in \omega$, such that $\sigma_i(0) \neq \sigma_j(0)$ when $i \neq j$. In particular, there are two interlacing members σ and τ of A with $\sigma(0) \neq \tau(0)$; we will use this fact often.

Proof. We proceed by induction on n. For $n = 1$, since A is uncountable, pick distinct members of A, σ_i for every $i \in \omega$. Assume that the lemma is true for $n = m$. Now let $n = m+1$. Let $B = pr^m A$. Then B is m-full. By the inductive hypothesis, there are mutually interlacing members of B, ρ_i for every $i \in \omega$, no two of which agree at 0. Now let $\alpha = \sup_{i \in \omega} \rho_i(m-1)$. For every $i \in \omega$, since $[\rho_i : A]$ is uncountable, find σ_i in $[\rho_i : A]$ with $\sigma_i(m) > \alpha$. This completes the induction. □

3.2.10: Lemma. Given ρ in ${}^{<\omega}\omega_1$, let A be (stationarily) n-full over ρ for some $n \geq 1$. Suppose $A = \bigcup_{i \in \omega} A_i$. Then A_i contains a set which is (stationarily) n-full over ρ for some $i \in \omega$.

Proof. Without loss of generality assume that $\rho = \emptyset$. We proceed by induction on n. For $n = 1$ the result is easy since a countable union of countable (nonstationary) subsets of ω_1 is countable (nonstationary). Assume the result is true for $n = m$. Now let $n = m+1$. Let $B = pr^m A$. Then B is (stationarily) m-full. For every $i \in \omega$, let $B_i = \{\sigma \in B \mid [\sigma : A_i]$ is (stationarily) 1-full over $\sigma\}$. For every σ in B, $[\sigma : A]$ is (stationarily) 1-full over σ, and so, by the case for $n = 1$, $[\sigma : A_i]$ is (stationarily) 1-full over σ for some $i \in \omega$. Thus $B = \bigcup_{i \in \omega} B_i$. By the inductive hypothesis, B_i contains a set which is (stationarily) m-full for some $i \in \omega$. For such an i, A_i contains a set which is (stationarily) n-full. This completes the induction. □

3.2.11: Lemma. Let A be a subset of P (Definition 3.2.1). Suppose that for every f in F there is a member of A which restricts f. Then A contains a set which is (stationarily) n-full for some $n \geq 1$.

Proof. Suppose not. By induction on m, define η_m in P_m for each $m \in \omega$ such that: (i) no subset of A is (stationarily) n-full over η_m for any $n \in \omega$ and (ii) $\eta_k \subset \eta_m$ when $k \leq m$. Notice that for every $m \in \omega$, $\eta_m \notin A$ since, in particular, no subset of A is (stationarily) 0-full over η_m.

First, let $\eta_0 = \emptyset$. Since A is contained in P, $\eta_0 \notin A$; i.e., no subset of A is (stationarily) 0-full over

η_0. Also, since no subset of A is (stationarily) n-full, by definition no subset of A is (stationarily) n-full over η_0 for any $n \geq 1$.

Now, given $m \in \omega$, having defined $\eta_0, \cdots, \eta_m$ satisfying the inductive requirements, define η_{m+1}. For each $n \in \omega$, since no subset of A is (stationarily) $(n+1)$-full over η_m, the set $A^{m+n+1}_{m+1} = \{\rho \in P_{m+1} \mid \rho$ extends η_m and some subset of A is (stationarily) n-full over $\rho\}$ is countable (not stationarily 1-full over η_m). Therefore the set $A_{m+1} = \bigcup_{n \in \omega} A^{m+n+1}_{m+1}$ is countable (not stationarily 1-full over η_m). So choose α_m in $\omega_1 - \{\rho(m) \mid \rho \in A_{m+1}\}$ such that $\alpha_m > \eta_m(m-1)$ if $m \geq 1$. Letting $\eta_{m+1} = \eta_m{}^\frown \alpha_m$, the functions $\eta_0, \cdots, \eta_{m+1}$ satisfy the inductive requirements. This completes the induction.

Now let $f = \bigcup_{m \in \omega} \eta_m$. For each $m \in \omega$, $f \upharpoonright m = \eta_m$, which is not in A. Thus f is in F but has no restriction in A, a contradiction. Therefore A contains a set which is (stationarily) n-full for some $n \geq 1$. □

<u>3.2.12: Lemma</u>. Let η be in ${}^{<\omega}\omega_1$. Let A be n-full over η for some $n \geq 1$, and let r_ρ be finite for every ρ in A. Then A has a subset B such that the family $(r_\rho : \rho \in B)$ is an n-full Δ-system over η.

<u>Proof.</u> Without loss of generality assume that $\eta = \emptyset$. We proceed by induction on n. For $n = 1$, this is just the usual Δ-system lemma for finite sets, but a direct proof is included for completeness.

Given a 1-full set A and a finite set r_ρ for each ρ in A, we must find a 1-full subset B of A such that the family $(r_\rho : \rho \in B)$ is a 1-full Δ-system. That is, B must be an uncountable subset of A such that the collection $\{r_\rho \mid \rho \in B\}$ is a Δ-system. By the pigeonhole principle, there is an $m \geq 1$ such that r_ρ has m elements for uncountably many members ρ of A. Therefore assume without loss of generality that r_ρ has m elements for every ρ in A. We proceed by induction on m. For $m = 1$ the result is easy. Assume the result is true for $m = k$. Now let $m = k+1$. There are two cases to consider.

Case I. There is an element x with $x \in r_\rho$ for uncountably many members ρ of A. Let $A' = \{\rho \in A \mid x \in r_\rho\}$. For each ρ in A' let $s_\rho = r_\rho - \{x\}$. By the inductive hypothesis there is a subset B of A' such that the family $(s_\rho : \rho \in B)$ is a 1-full Δ-system. Then the family $(r_\rho : \rho \in B)$ is also a 1-full Δ-system.

Case II. There is no such element x. Recursively choose $\rho(\alpha)$ in A for each $\alpha < \omega_1$ such that

$r_{\rho(\alpha)} \cap r_{\rho(\xi)} = \emptyset$ for every $\xi < \alpha$. Let $B = \{\rho(\alpha) \mid \alpha < \omega_1\}$. Then the family $(r_\rho : \rho \in B)$ is a 1-full Δ-system.

This completes the induction needed to prove the lemma for $n = 1$.

Assume that the lemma is true for $n = m$. Now let $n = m+1$. Let $D = pr^m A$, and let $C_\sigma = [\sigma : A]$ for each σ in D. Then D is m-full and C_σ is 1-full over σ for each σ in D. By the usual Δ-system lemma, each C_σ contains an uncountable subset C'_σ such that the collection $\{r_\rho \mid \rho \in C'_\sigma\}$ is a Δ-system with root t_σ. Applying the inductive hypothesis to the family $(t_\sigma : \sigma \in D)$, we obtain a subset D' of D such that the family $(t_\sigma : \sigma \in D')$ is an m-full Δ-system with root family $(t_\theta : \theta \in prD')$. The set $C' = \bigcup\{C'_\sigma \mid \sigma \in D'\}$ is almost the desired set B, but it must be thinned down a little. For any distinct members σ and τ of $pr^m B$, $r_\rho \cap r_\theta$ is to be independent of the choice of ρ in $[\sigma : B]$ and θ in $[\tau : B]$.

Enumerate D' as $\{\tau(\alpha) \mid \alpha < \omega_1\}$ with each member of D' counted ω_1 times. For each σ in D', well order C'_σ. Now, for every $\alpha < \omega_1$, recursively define $\rho(\alpha) \in C'$ along with $\sigma(\alpha) = \rho(\alpha) \upharpoonright m$. Having already defined $\rho(\xi)$ and $\sigma(\xi)$ for $\xi < \alpha$, let $U_\alpha = \bigcup_{\xi < \alpha} (r_{\rho(\xi)} - t_{\sigma(\xi)})$. If $t_{\tau(\alpha)} \cap U_\alpha = \emptyset$, then let $\sigma(\alpha) = \tau(\alpha)$. Otherwise, let j be the largest integer i such that $t_{\tau(\alpha) \upharpoonright i} \cap U_\alpha = \emptyset$. Let $\sigma(\alpha)$ be the first $\sigma \in D'$ (in the given enumeration) such that σ extends $\tau(\alpha) \upharpoonright j$, $\sigma \upharpoonright (j+1)$ has not yet occurred as $\sigma(\xi) \upharpoonright (j+1)$ for any $\xi < \alpha$, and $t_\sigma \cap U_\alpha = \emptyset$. Now let $\rho(\alpha)$ be the first $\rho \in C'_{\sigma(\alpha)}$

such that: (i) $\rho \neq \rho(\xi)$ for any $\xi < \alpha$, and (ii) $(r_\rho - t_{\sigma(\alpha)}) \cap \bigcup_{\xi < \alpha} r_{\rho(\xi)} = \emptyset$. Now let $B = \{\rho(\alpha) \mid \alpha < \omega_1\}$. Then the family $(r_\rho : \rho \in B)$ is an $(m+1)$-full Δ-system with root family $(t_\theta : \theta \in prB)$ where $t_\theta = r_\theta$ for $\theta \in B$. This completes the induction needed to prove the lemma. $\square$

Section 4.1. A Para-Lindelöf, Nonnormal Moore Space Constructed Under $MA(\omega_1)$.

In this section the ideas of Chapter 3 are used with a normal ω_1-Cantor tree as a reference space in constructing a new example. While ω_1-Cantor trees are not collectionwise Hausdorff, they are Moore spaces which, if $MA(\omega_1)$ is assumed, are normal.

A tree [3] is a partially ordered set $(T, \leq)$ such that, for every t in T, the set $\{x \in T \mid x < t\}$ of all strict predecessors of t is well-ordered by $<$. For any ordinal α, the αth level of T is $\{t \in T \mid$ the set of all predecessors of t under $<$ has order type $\alpha\}$. The height of T is the smallest ordinal α such that the αth level of T is empty. An open interval J in T is of the form $\{t\}$ for some t in the 0th level of T or $\{x \in T \mid s < x \leq t\}$ for some s and t in T. The tree topology on T, which is always Hausdorff and regular (i.e., T_3), is the topology generated by the base of all open intervals J in T.

The Cantor tree C is the tree $^{\leq \omega}2$ of the height $\omega + 1$, partially ordered by inclusion and topologized by the tree topology. Let C' be the ωth level of C — that is, $^{\omega}2$. If $\omega \leq \kappa \leq c$, then a κ-Cantor tree B is a subset of C of cardinality κ which contains all of $C - C'$.

<u>4.1.1: Theorem</u> [3]. If $\omega < \kappa < c$, let B be a κ-Cantor tree. Then:

(a) B is a separable, locally compact Moore space which is not collectionwise Hausdorff.

(b) (MA(κ)) B is normal. $\square$

<u>4.1.2: Notational Definition</u> (MA(ω_1)). Let T be an ω_1-Cantor tree. Let T' be the ωth level of T, a discrete subset of T. Given $n \geq 1$, since $|P_n| = \omega_1$, let $s_n : P_n \to T'$ be a one-one, onto function assigning to each σ in P_n the sequence $s_n(\sigma)$ of 0's and 1's. Since T is normal under MA(ω_1), for each subset of A of P_N, there is a function $d_A : P_n \to \omega$ such that $s_n(\sigma) \restriction m \neq s_n(\tau) \restriction m$ when $\sigma \in A$, $\tau \in P_n - A$, and $m = \max(d_A(\sigma), d_A(\tau))$. $\square$

<u>4.1.3: Notational Definition</u>. Recall (Definition 3.2.1) that $F = \{f : \omega \to \omega_1 \mid f \text{ is increasing}\}$. Now let $H = \{f^\frown 0 \mid f \in F\}$, $K = \{f^\frown 1 \mid f \in F\}$. Thus H and K are two disjoint copies of F. $\square$

<u>4.1.4: Notational Definition</u>: Let ρ be in ${}^n\omega_1$ for some $n \geq 1$. Let $[\rho : F] = \{f \in F \mid f \restriction n = \rho\}$. Define $[\rho : H]$ and $[\rho : K]$ analogously. $\square$

<u>4.1.5: Example</u> (MA(ω_1)). Given $n \geq 1$, let $I_n = \{(\sigma, \tau) \in P_n^2 \mid$ (i) $\sigma(0) \neq \tau(0)$, (ii) σ and τ interlace, and (iii) $s_i(\sigma \restriction i) \restriction n = s_i(\tau \restriction i) \restriction n$ for $1 \leq i \leq n\}$; Let $I_n^* = \bigcup_{m \geq n} I_m$. Now let $I = I_1^*$ and $X = H \cup K \cup I$.

Let X be topologized as follows. Given x in I, $\{x\}$

is open; that is, the points of I are isolated. Given $n \geq 1$ and h in H, the set $U_n(h) = [h \upharpoonright n : H] \cup \{(\sigma, \tau) \in I_n^* \mid \sigma \upharpoonright n = h \upharpoonright n\}$ is open. Similarly, given $n \geq 1$ and k in K, the set $U_n(k) = [k \upharpoonright n : K] \cup \{(\sigma, \tau) \in I_n^* \mid \tau \upharpoonright n = k \upharpoonright n\}$ is open. These open sets make up the base $\mathcal{B}$ for the topology on X.

The topological space X has the following properties:

(a) X is T_1 and zero-dimensional and hence also regular and in fact completely regular.

(b) X is a Moore space with a σ-disjoint, σ-locally countable base.

(c) X is para-Lindelöf.

(d) X is not normal.

(e) X is metacompact.

(f) X is countably paracompact.

Frequently, a verification involving points in H and the corresponding one involving points in K are totally analogous. In such cases, without any further mention, only one verification will be given.

(a): First, we see that X is T_1; i.e., singletons are closed. For each x in X, $\{x'\}$ separates any distinct point x' in I from x. Given a point x in I_n, any point h in H can be separated from x by $U_{n+1}(h)$. Now consider a point h in H. Any other point h' in H can be separated from h by $U_n(h')$

where n is large enough so that $h' \upharpoonright n \neq h \upharpoonright n$. Also, for any k in K, even $U_1(k)$ separates k from h.

In addition, the given basic open sets are closed. Each $\{x\}$ contained in I is closed as X is T_1. Now, given h in H and $n \geq 1$, consider $U = U_n(h)$. Each x in $I - U$ is separated from U by $\{x\}$. Given h' in $H - U$, $U_m(h')$ separates h' from U where m is large enough so that $h' \upharpoonright m \neq h \upharpoonright m$ (e.g., $m = n$). In separating k in K from U, there are two possibilities. If $k(0) = h(0)$, then $U_1(k)$ and U are disjoint. Otherwise, since $k \upharpoonright 1 \neq h \upharpoonright 1$, find m large enough so that $s_1(k \upharpoonright 1) \upharpoonright m \neq s_1(h \upharpoonright 1) \upharpoonright m$; then $U_m(k)$ and U are disjoint. Thus the given base $\mathcal{B}$ is clopen; so X is zero-dimensional.

Since X is T_1 and has a clopen base, it is T_3 and in fact Tychonoff.

(b): Let $n \geq 1$ be given. Let $\mathcal{G}_n^0 = \{U_n(h) \mid h \in H\}$, $\mathcal{G}_n^1 = \{U_n(k) \mid k \in K\}$, and $\mathcal{G}_n^2 = \{\{x\} \mid x \in I - \bigcup(\mathcal{G}_n^0 \cup \mathcal{G}_n^1)\}$. Let $\mathcal{G}_n = \mathcal{G}_n^0 \cup \mathcal{G}_n^1 \cup \mathcal{G}_n^2$. Each collection $\mathcal{G}_n^e$ is pairwise disjoint for $e = 0, 1,$ or 2. Since the base $\mathcal{B} = \bigcup_{n \geq 1} \mathcal{G}_n$, $\mathcal{B}$ is a σ-disjoint base. Since the collection $\{\mathcal{G}_n \mid n \geq 1\}$ of open covers of X is a development for X, X is a Moore space.

Finally, we verify that $\mathcal{B}$ is σ-locally countable since each $\mathcal{G}_n$ is locally countable as witnessed by $\mathcal{G}_{n+1}$. Let $n \geq 1$ be given. Clearly, for each x in I, $\{x\}$ meets at most two members of $\mathcal{G}_n$. Also, for each x in $H \cup K$, $U_{n+1}(x)$ meets only countably many members of $\mathcal{G}_n$. To verify this, consider $U = U_{n+1}(h)$ where $h \in H$. U meets exactly one member of $\mathcal{G}_n^0$ — namely, $U_n(h)$ — and no members of $\mathcal{G}_n^2$. Now suppose U meets $U_n(k)$ where $k \in K$. Choose (σ, τ) in $U \cap U_n(k)$. Then $h \upharpoonright (n+1) = \sigma \upharpoonright (n+1)$ interlaces with $\tau \upharpoonright (n+1)$. So there are only countably many possibilities for $\tau \upharpoonright n = k \upharpoonright n$, which determines $U_n(k)$ for k in K. Thus U meets only countably many members of $\mathcal{G}_n^1$ and hence of $\mathcal{G}_n$.

(c): To verify that X is para-Lindelöf, let $\mathcal{V}$ be an open cover of X. For each f in F, let $c(f)$ be the smallest integer n such that both $U_n(f^\frown 0)$ and $U_n(f^\frown 1)$ are contained in members of $\mathcal{V}$. Given $n \geq 1$, let $A_n = \{\rho \in P_n \mid c(f) = n$ for some (every) f in F which extends $\rho\}$, or equivalently $A_n = \{f \upharpoonright n \mid f \in F$ and $c(f) = n\}$; let $d_n : P_n \to \omega$ be a function separating A_n from $P_n - A_n$ as in Definition 4.1.2. Finally, for each f in F, let $d(f)$ be the larger of $c(f)$ and $\max_{i \leq c(f)} d_i(f \upharpoonright i)$. For convenience, extend the functions c and d from F to $F \cup H \cup K$ by letting $c(x) = c(x \upharpoonright \omega)$ and $d(x) = d(x \upharpoonright \omega)$ for each x in $H \cup K$.

For each x in $H \cup K$, let $W(x) = U_{c(x)}(x) - \{(\sigma, \tau) \in I^*_{c(x)} \mid s_i(\sigma \upharpoonright i) \upharpoonright d(x) \neq s_i(\tau \upharpoonright i) \upharpoonright d(x)$ for some $i \leq c(x)\}$. To verify that each $W(x)$ is open (and hence clopen), consider $W(h)$ where $h \in H$. Clearly $W(h) \cap I$ is open. For each h'

in H, $h' \in W(h)$ iff $h' \upharpoonright c(h) = h \upharpoonright c(h)$. Moreover, if $h' \in W(h) \cap H$, then $U_{d(h)}(h') \subset W(h)$. Also notice that for h in H, $h \upharpoonright c(h)$ determines $c(h)$, $d(h)$ and $W(h)$; the same holds for k in K. Now let $\mathfrak{w}^0 = \{W(h) \mid h \in H\}$, $\mathfrak{w}^1 = \{W(k) \mid k \in K\}$, and $\mathfrak{w}^2 = \{\{x\} \mid x \in I - \bigcup(\mathfrak{w}^0 \cup \mathfrak{w}^1)\}$. Let $\mathfrak{w}$ be the open cover $\mathfrak{w}^0 \cup \mathfrak{w}^1 \cup \mathfrak{w}^2$, which refines $\mathfrak{v}$.

To see that $\mathfrak{w}$ is in fact locally countable, we first investigate the amount of intersection between its members. Both $\mathfrak{w}^0 \cup \mathfrak{w}^2$ and $\mathfrak{w}^1 \cup \mathfrak{w}^2$ are pairwise disjoint collections. Now suppose $W(h)$ meets $W(k)$ where $h \in H$ and $k \in K$. I claim that $c(h) = c(k)$. To verify this, suppose that $c(h) \neq c(k)$. Without loss of generality assume $c(h) < c(k)$, and let $n = c(h)$. Choose (σ, τ) in $W(h) \cap W(k)$. Then $\sigma \upharpoonright n \in A_n$ but $\tau \upharpoonright n \in P_n - A_n$. So $s_n(\sigma \upharpoonright n) \upharpoonright m \neq s_n(\tau \upharpoonright n) \upharpoonright m$ where $m = \max(d_n(\sigma \upharpoonright n), d_n(\tau \upharpoonright n))$. If $m = d_n(\sigma \upharpoonright n)$, then $m \leq d(h)$, and so $(\sigma, \tau) \notin W(h)$. Otherwise, $m = d_n(\tau \upharpoonright n)$, $m \leq d_n(\tau \upharpoonright n)$, $m \leq d(k)$, and so $(\sigma, \tau) \notin W(k)$. This is a contradiction, and therefore $c(h) = c(k)$.

We now verify that $\mathfrak{w}$ is locally countable. Given any x in I, $\{x\}$ meets at most two members of $\mathfrak{w}$. For each x in $H \cup K$, let $Z(x)$ be the open set $W(x) \cap U_{c(x)+1}(x)$. In checking that each $Z(x)$ avoids all but countably many members of $\mathfrak{w}$, consider $Z(h)$ where $h \in H$ and $c(h) = n$. $Z(h)$ meets just one member of $\mathfrak{w}^0 \cup \mathfrak{w}^2$ — namely, $W(h)$. Because $Z(h)$ refines $W(h)$, every member of $\mathfrak{w}^1$ which $Z(h)$ meets lies in $\mathfrak{w}^1_n$ — where $\mathfrak{w}^1_n = \{W(k) \mid k \in K$ and $c(k) = n\}$ — and is therefore contained in a member of $\mathcal{G}_n^1$. But since $Z(h)$ also refines $U_{n+1}(h)$, it meets only countably many members of $\mathcal{G}_n^1$, each of which contains at most one member of $\mathcal{W}_n^1$. So $Z(h)$ meets just countably many members of $\mathcal{W}^1$ and hence of $\mathcal{W}$. This completes the proof that $\mathcal{W}$ is locally countable and that X is therefore para-Lindelöf.

(d): To verify that X is not normal, let V_1 and V_2 be arbitrary open sets which contain the disjoint closed sets H and K, respectively. We will see that V_1 and V_2 intersect and thus that H and K cannot be separated by disjoint open sets. With $\mathcal{V}$ the open cover $\{V_1, V_2, I\}$ of X, let $c(f)$ for each f in F and A_n for each $n \geq 1$ be as in the proof of (c). Let $A = \bigcup_{n \geq 1} A_n$. Since every f in F has a restriction in A, by Lemma 3.2.11, choose $n \geq 1$ so that A_n has an n-full subset, and let B be such a subset. Define the "matching" function $M : B \rightarrow ({}^n2)^n$ (that is, the set of n-tuples on ${}^n2$) by letting $M(\rho)$ be the n-tuple $(s_1(\rho \upharpoonright 1) \upharpoonright n, \cdots, s_n(\rho \upharpoonright n)$ each ρ in B. Since there are only finitely many possibilities for $M(\rho)$, by Lemma 3.2.10, let C be an n-full subset of B on which M is constant. Now, by Lemma 3.2.9, find two interlacing members σ and τ of C with $\sigma(0) \neq \tau(0)$. Then the point (σ, τ) lies in $V_1 \cap V_2$. Therefore X is not normal.

(e): We now verify that X is metacompact. Given any open cover $\mathcal{V}$ of X, the locally countable open refinement $\mathcal{W}$

constructed in the proof of (c) is also point-finite. However, while the ω_1-Cantor tree sequences are necessary in proving that X is para-Lindelöf, they are not important here. We can just as well find a point-finite open refinement $\mathfrak{U}$ for $\mathfrak{V}$ more simply. Define the function c on $F \cup H \cup K$ as in the proof of (c). Then let $\mathfrak{U}^0 = \{U_{c(h)}(h) \mid h \in H\}$, $\mathfrak{U}^1 = \{U_{c(k)}(k) \mid k \in K\}$, and $\mathfrak{U}^2 = \{\{x\} \mid x \in I - \bigcup(\mathfrak{U}^0 \cup \mathfrak{U}^1)\}$. Let $\mathfrak{U}$ be the open cover $\mathfrak{U}^0 \cup \mathfrak{U}^1 \cup \mathfrak{U}^2$, refining $\mathfrak{V}$. Both $\mathfrak{U}^0 \cup \mathfrak{U}^2$ and $\mathfrak{U}^1 \cup \mathfrak{U}^2$ are pairwise disjoint collections. Since each point in $H \cup K$ lies in exactly one member of $\mathfrak{U}$ and each point in I lies in at most two, $\mathfrak{U}$ is indeed point-finite. Thus X is metacompact. Notice that in this proof we could have defined the function c independently on H and K.

(f): Finally, to verify that X is countably paracompact, let $\mathfrak{V}$ be a countably infinite open cover of X. Index $\mathfrak{V}$ as $\{V_i \mid i \geq 1\}$. For each f in F, let $c(f)$ be the smallest integer n such that both $U_n(f^\frown 0)$ and $U_n(f^\frown 1)$ are contained in sets V_i with $i \leq n$. With $c(f)$ so defined for each f in F, go on to define A_n and d_n for each $n \geq 1$; $d(f)$ for each f in F; and $c(x)$, $d(x)$, and $W(x)$ for each x in $H \cup K$ accordingly as in the proof of (c). Also let $\mathfrak{W}^0$, $\mathfrak{W}^1$, and $\mathfrak{W}^2$ be as in the proof of (c). As before, each $W(x)$ is open, and $W(x)$ cannot meet $W(x')$ where x and x' are in $H \cup K$, unless $c(x) = c(x')$.

For each $n \geq 1$, let $W_n^* = \cup\{W(x) \mid x \in H \cup K$ and $c(x) = n\} \cup \{x \in I \mid \{x\} \in \mathfrak{w}^2$ and $x \in V_n - \bigcup_{i<n} V_i\}$. Then the open cover $\mathfrak{w}^* = \{W_n^* \cap V_i \mid n \geq 1$ and $i \leq n\}$ is a locally finite refinement for $\mathfrak{V}$. In fact, $\mathfrak{w}^*$ is star-finite; that is, each member of $\mathfrak{w}^*$ avoids all but finitely many members of $\mathfrak{w}^*$. □

In this example, the ω_1-Cantor tree sequences are essential in getting X to be para-Lindelöf and also countably paracompact. For every $i \geq 1$, the set P_i is identified with the top level T' of the ω_1-Cantor tree T by the function s_i.

Let h in H and $n \geq 1$ be given. Any "nth level" open neighborhood of h -- that is, one whose intersection with $H \cup K$ is $[h \upharpoonright n : H]$ and whose intersection with I is determined by its intersection with I_n — can be thinned down by an association with the open neighborhoods of $s_i(h \upharpoonright i)$ in T for each $i \leq n$. More precisely, the nth level open set $U_n(h)$ is associated with the open interval $\{t \in T \mid s_i(h \upharpoonright i) \upharpoonright n \leq t \leq s_i(h \upharpoonright i)\}$ in T for each $i \leq n$. But if $m > n$, a smaller nth level open neighborhood of h, associated with the open interval $\{t \in T \mid s_i(h \upharpoonright i) \upharpoonright m \leq t \leq s_i(h \upharpoonright i)\}$ in T for each $i \leq n$, is $U_n(h) - \{(\sigma, \tau) \in I_n^* \mid s_i(\sigma \upharpoonright i) \upharpoonright m \neq s_i(\tau \upharpoonright i) \upharpoonright m$ for some $i \leq n\}$. Similarly, let k in K and $n \geq 1$ be given. Any nth level open neighborhood of k — that is, one whose intersection with $H \cup K$ is $[k \upharpoonright n : K]$ and whose intersection with I is determined by its intersection with I_n — can be thinned down by an association with the open neighborhoods of

$s_i(k \restriction i)$ in T for each $i \leq n$.

Because the ω_1-Cantor tree T is normal, this thinning down provides enough separation to make X para-Lindelöf. For it gives a handle for controlling the "levels" of intersecting open sets. At the same time, T is not collectionwise Hausdorff since no uncountable subset of the discrete set T' can be separated by disjoint open sets. Therefore, certainly, the finite sequences $s_i(\rho \restriction i) \restriction n$ with $i \leq n$ cannot distinguish uncountably many members ρ of any given P_n from each other; moreover, uncountably many of them will agree on all the sequences $s_i(\rho \restriction i) \restriction m$ with $i \leq n$, no matter how large the integer m is. In view of this, the thinning down provides little enough separation so that X remains nonnormal.

Section 4.2. A Normal, Para-Lindelöf, Collectionwise Nonnormal Moore Space Constructed Under $MA(\omega_1)$.

Now, by embedding the techniques used in the previous section in a slightly different picture, we construct another example. In this topological space there is only one copy of F, and the pairs of interlacing partial functions, which provide interesting intersections between open sets, are unordered pairs rather than ordered pairs.

<u>4.2.1: Example</u> $(MA(\omega_1))$. Given $n \geq 1$, let $J_n = \{\{\sigma, \tau\} \subset P_n \mid$ (i) $\sigma(0) \neq \tau(0)$, (ii) σ and τ interlace, and (iii) $s_i(\sigma \restriction i) \restriction n = s_i(\tau \restriction i) \restriction n$ for $1 \leq i \leq n\}$; let $J_n^* = \bigcup_{m \geq n} J_m$. Now let $J = J_1^*$ and $Y = F \cup J$.

Let Y be topologized as follows. Given y in J, $\{y\}$ is open; that is, the points of J are isolated. Given $n \geq 1$ and f in F, the set $U_n(f) = [f \restriction n : F] \cup \{\{\sigma, \tau\} \in J_n^* \mid \sigma \restriction n = f \restriction n$ or $\tau \restriction n = f \restriction n\}$ is open. These open sets make up the base $\mathcal{B}$ for the topology on Y.

The topological space Y has the following properties:

(a) Y is T_1 and zero-dimensional and also normal.

(b) Y is a Moore space with a σ-locally countable base but but with no σ-disjoint base.

(c) Y is para-Lindelöf.

(d) Y is not collectionwise normal.

(e) Y is metacompact.

(f) Y is countably paracompact.

In the discussion of this example, the topological space X referred to is that of Example 4.1.5.

(a): Proving that Y is T_1 and that the base $\mathcal{B}$ for Y is clopen is very much like proving the analogous result for X. For example, consider the basic open set $U = U_n(f)$ where $n \geq 1$ and $f \in F$ we verify that U is closed. If $y \in J - U$, then $\{y\}$ separates y from U. In separating f' in $F - U$ from U, there are two possibilities. If $f'(0) = f(0)$, then $U_m(f') \cap U$ is empty where m is large enough so that $f' \restriction m \neq f \restriction m$ (e.g., $m = n$). Otherwise, since $f' \restriction 1 \neq f \restriction 1$, find m large enough so that $s_1(f' \restriction 1) \restriction m \neq s_1(f \restriction 1) \restriction m$; then $U_m(f') \cap U$ is empty.

To verify that Y is normal, let A and B be disjoint closed subsets of Y. Without loss of generality we assume that both A and B are contained in F. For, if not, let $A' = A \cap F$ and $B' = B \cap F$. If U' and V' are disjoint open sets containing A' and B', respectively, then the open sets $U = U' - B \cup (A \cap J)$ and $V = V' - A \cup (B \cap J)$ are disjoint and contain A and B, respectively.

Now, for each f in A, let $c(f)$ be the smallest integer n such that $U_n(f) \cap B$ is empty. Similarly, for each f in B, let $c(f)$ be the smallest integer n such that $U_n(f) \cap A$ is empty. Let $n \geq 1$ be given. Let $A_n = \{f \restriction n \mid f \in A$ and $c(f) = n\}$ and $B_n = \{f \restriction n \mid f \in B$ and $c(f) = n\}$. Let $d_n : P_n \to \omega$ be a function separating A_n from $P_n - A_n$ and B_n from $P_n - B_n$; for example, let d_n be the maximum of two functions d_{A_n} and d_{B_n}

guaranteed to exist as discussed in Definition 4.1.2. Finally, given f in $A \cup B$, let $d(f)$ be the larger of $c(f)$ and $\max_{i \leq c(f)} d_i(f \upharpoonright i)$. Let $U = \bigcup \{U_{d(f)}(f) \mid f \in A\}$ and $V = \{U_{d(f)}(f) \mid f \in B\}$. Then U and V are open sets which contain A and B, respectively.

Certainly no point of F can be in both U and V since $d(f) \geq c(f)$ for each f in $A \cup B$. To see that U and V are disjoint, suppose that $\{\sigma, \tau\} \in U_{d(f)}(f) \cap U_{d(f')}(f')$ for some f in A and f' in B. We assume without loss of generality that $c(f) \leq c(f')$, $\sigma \upharpoonright c(f) = f \upharpoonright c(f)$, and $\tau \upharpoonright c(f') = f' \upharpoonright c(f')$. Let $n = c(f)$. Then $\sigma \upharpoonright n \in A_n$ but $\tau \upharpoonright n \in P_n - A_n$. Thus $s_n(\sigma \upharpoonright n) \upharpoonright m \neq s_n(\tau \upharpoonright n) \upharpoonright m$ where $m = \max(d_n(\sigma \upharpoonright n), d_n(\tau \upharpoonright n))$. If $m = d_n(\sigma \upharpoonright n)$, then $m \leq d(f)$, and so $\{\sigma, \tau\} \notin U_{d(f)}(f)$. Otherwise, $m = d_n(\tau \upharpoonright n)$, $m \leq d(f')$, and so $\{\sigma, \tau\} \notin U_{d(f')}(f')$. This is impossible, and therefore U and V are disjoint. So Y is T_4.

(b): Proving that Y is a Moore space with a σ-locally countable base is very similar to proving the same result for X. Given $n \geq 1$, let $\mathcal{G}_n^1 = \{U_n(f) \mid f \in F\}$, $\mathcal{G}_n^2 = \{\{y\} \mid y \in J - \bigcup \mathcal{G}_n^1\}$, and $\mathcal{G}_n = \mathcal{G}_n^1 \cup \mathcal{G}_n^2$. Since the collection $\{\mathcal{G}_n \mid n \geq 1\}$ of open covers of Y is a development for Y, Y is a Moore space. Also, each $\mathcal{G}_n$ is locally countable as witnessed by $\mathcal{G}_{n+1}$; the argument verifying this matches the one used to verify the analogous result for X. Thus the base $\mathcal{B}$ is σ-locally countable since $\mathcal{B} = \bigcup_{n \geq 1} \mathcal{G}_n$. Unlike X, however, Y does not have a σ-disjoint base; this is shown by the method used to prove that Y is not collectionwise normal.

(c): The proof that Y is para-Lindelöf follows the lines of the proof that X is para-Lindelöf. Let $\mathcal{V}$ be an open cover of Y. For each f in F, let $c(f)$ be the smallest integer n such that $U_n(f)$ is contained in a member of $\mathcal{V}$. As before, for each $n \geq 1$, let $A_n = \{f \upharpoonright n \mid f \in F \text{ and } c(f) = n\}$; let $d_n : P_n \to \omega$ be a function separating A_n from $P_n - A_n$ as in Definition 4.1.2. For each f in F, let $d(f)$ be the larger of $c(f)$ and $\max_{i \leq c(f)} d_i(f \upharpoonright i)$.

For each f in F, let $W(f) = U_{c(f)}(f) - \{\{\sigma, \tau\} \in J^*_{c(f)} \mid s_i(\sigma \upharpoonright i) \upharpoonright d(f) \neq s_i(\tau \upharpoonright i) \upharpoonright d(f)$ for some $i \leq c(f)\}$. Now let $\mathcal{W}^1 = \{W(f) \mid f \in F\}$, $\mathcal{W}^2 = \{\{y\} \mid y \in J - \bigcup \mathcal{W}^1\}$, and finally $\mathcal{W} = \mathcal{W}^1 \cup \mathcal{W}^2$. Following the proof for X, we see that each $W(f)$, which is determined by $f \upharpoonright c(f)$, is open and that for any f and f' in F, $W(f)$ cannot meet $W(f')$ unless $c(f) = c(f')$. We see as well that the open cover $\mathcal{W}$, which refines $\mathcal{V}$, is locally countable; for each f in F, the open set $Z(f) = W(f) \cap U_{c(f)+1}(f)$ witnesses the local countability of $\mathcal{W}$ at f.

(d): The proof that Y is not collectionwise normal resembles the proof that X is not normal. For each $\alpha < \omega_1$, let $F_\alpha = \{f \in F \mid f(0) = \alpha\}$. We check first that each F_α is closed and that the collection $\mathfrak{F} = \{F_\alpha \mid \alpha < \omega_1\}$ is discrete. Given any y in J, $\{y\}$ meets no member of $\mathfrak{F}$. Given any f in F, let

$\alpha = f(0)$; then $f \in F_\alpha$, which is the only member of $\mathfrak{F}$ met by $U_1(f)$.

To verify that the collection $\mathfrak{F}$ cannot be separated by disjoint open sets, let $\{V_\alpha \mid \alpha < \omega_1\}$ be an arbitrary collection of open sets with $V_\alpha \cap F = F_\alpha$ for every $\alpha < \omega_1$. We will see that this collection is not pairwise disjoint. With $\mathcal{V}$ the open cover $\{V_\alpha \mid \alpha < \omega_1\} \cup \{J\}$ of Y, let $c(f)$ for each f in F and A_n for each $n \geq 1$ be as in the proof of (c). Now let $n \geq 1$, $B \subset A_n$, $M : B \to ({}^n 2)^n$, and $C \subset B$ be as in the proof that X is not normal. By Lemma 3.2.9, find two interlacing members σ and τ of C with $\sigma(0) \neq \tau(0)$. Then the point $\{\sigma, \tau\}$ lies in $V_{\sigma(0)} \cap V_{\tau(0)}$. Therefore Y is not collectionwise normal.

(e): The proof that Y is metacompact is just like the analogous proof for X. Given an open cover $\mathcal{V}$ of Y, the locally countable open refinement $\mathcal{W}$ constructed in proving (c) is also point-finite. More simply, however, with the function c on F defined as in the proof of (c), let $\mathcal{U}^1 = \{U_{c(f)}(f) \mid f \in F\}$, $\mathcal{U}^2 = \{\{y\} \mid y \in J - \bigcup \mathcal{U}^1\}$, and $\mathcal{U} = \mathcal{U}^1 \cup \mathcal{U}^2$. Then $\mathcal{U}$ is a point-finite open refinement for $\mathcal{V}$; each point in F lies in just one member of $\mathcal{U}$, and each point in J lies in at most two.

(f): Just as we did for X, we now modify the proof that Y is para-Lindelöf to prove that Y is countably paracompact. Given a countably infinite open cover $\mathcal{V} = \{V_i \mid i \geq 1\}$ of Y, for each f in F, let $c(f)$ be the smallest integer n such that $U_n(f)$ is is contained in a set V_i with $i \leq n$. Define A_n and d_n for each $n \geq 1$; $d(f)$ and $W(f)$ for each f in F; and $\mathfrak{w}^1$ and $\mathfrak{w}^2$ accordingly as in the proof of (c). Once again, each $W(f)$ is open, and $c(f) = c(f')$ if $W(f)$ meets $W(f')$ for any f and f' in F.

For each $n \geq 1$, let $W_n^* = \bigcup\{W(f) \mid f \in F \text{ and } c(f) = n\} \cup \{y \in J \mid \{y\} \in \mathfrak{w}^2 \text{ and } y \in V_n - \bigcup_{i<n} V_i\}$. The open cover $\mathfrak{w}^* = \{\mathfrak{w}_n^* \cap V_i \mid n \geq 1 \text{ and } i \leq n\}$ is a locally finite (in fact star-finite) refinement for $\mathfrak{v}$. □

Section 5.1. A Para-Lindelöf, Nonnormal T_3-Space.

In this section we construct a space X like that of Section 4.1. For the two examples in this chapter, we use Bing's G as a reference space, just as we used a normal ω_1-Cantor tree in Chapter 4. Without any extra axioms for set theory, Bing's G is a T_4 space which is not collectionwise Hausdorff. Therefore the constructions in this chapter do not require any special set-theoretic assumptions. However, like Bing's G, both examples in this chapter have character 2^{ω_1}.

5.1.1: Definition of Bing's G [3]. Let $S = \mathcal{P}(\omega_1)$ and $G = {}^{S}2$, the set of all functions from S into $\{0,1\}$. For any $\alpha < \omega_1$, let $g_\alpha : S \to 2$ be the characteristic function for α; that is, for any $A \subset \omega_1$, $g_\alpha(A) = 1$ iff $\alpha \in A$. Let $G' = \{g_\alpha \mid \alpha < \omega_1\}$. To topologize G, let all the points of $G - G'$ be isolated. Also, given any $A \subset \omega_1$ and $e = 0$ or 1, the set $U_{A,e} = \{g \in G \mid g(A) = e\}$ is open. The singletons on $G - G'$ together with all the open sets $U_{A,e}$ provide a subbase for the topology on G. □

5.1.2: Theorem [3]. Bing's G is T_4 but not collectionwise Hausdorff; no uncountable subset of the discrete set G' can be separated by disjoint open sets. □

5.1.3: Notational Definition. Given any $n \geq 1$, let $S_n = \mathcal{P}(P_n)$, and let T_n be the set of all finite subsets of $\bigcup_{i \leq n} S_i$. Let F, H, and K be as in Chapter 4. Also let $[\rho : F]$, $[\rho : H]$, $[\rho : K]$ be as in Chapter 4, where $\rho \in {}^{<\omega}\omega_1$. □

<u>5.1.4: Example</u>. For each $n \geq 1$, let $I_n = \{(\sigma, \tau, r) \in P_n^2 \times T_n \mid$ (i) $\sigma(0) \neq \tau(0)$, and (ii) σ and τ interlace$\}$; let $I_n^* = \bigcup_{m \geq n} I_m$. Now let $I = I_1^*$ and $X = H \cup K \cup I$.

Now we topologize X. The points of I are isolated. For each $n \geq 1$, h in H, and t in T_n, let $U_n(h,t) = [h \restriction n : H] \cup \{(\sigma, \tau, r) \in I_n^* \mid$ (i) $\sigma \restriction n = h \restriction n$, and (ii) given any $E \in t \cap S_i$ for some $i \leq n$, $E \in r$ iff $h \restriction i \in E\}$; these sets are open. Similarly, for each $n \geq 1$, k in K, and t in T_n, let $U_n(k,t) = [k \restriction n : K] \cup \{(\sigma, \tau, r) \in I_n^* \mid$ (i') $\tau \restriction n = k \restriction n$, and given any $E \in t \cap S_i$ for some $i \leq n$, $E \in r$ iff $k \restriction i \in E\}$; these sets too are open. The singletons on I together with these open sets make up the base $\mathcal{B}$ for X. If $m \leq n$, notice that the intersection of the basic open sets $U_m(x,t)$ and $U_n(x,t')$ is $U_n(x, t \cup t')$, and $U_m(x,t)$ contains $U_n(x,t')$ if $t \subset t'$.

The topological space X has the following properties:

(a) X is T_1 and zero-dimensional and therefore also regular and furthermore completely regular.

(b) X has character 2^{ω_1}; therefore X is not a Moore space and does not have a σ-disjoint or a σ-locally countable base.

(c) X is para-Lindelöf.

(d) X is not normal.

(e) X is metacompact.

(f) X is countably paracompact.

In the discussion of this example, $X(4)$ will refer to the space X of Example 4.1.5. As in the discussion of $X(4)$, two corresponding verifications involving points of H and of K, respectively, will often be represented by just one of them.

(a): X is T_1 exactly as $X(4)$ is; just replace any basic open set $U_n(x)$ in $X(4)$ by the basic open set $U_n(x, \emptyset)$ in X. To see that X is zero-dimensional, we verify that any basic open set $U = U_n(h, t)$, where $h \in H$ and $\{h \upharpoonright 1\} \in t$, is closed. The points of $I - U$ and of $H - U$ can be separated from U just as in $X(4)$ — using $U_m(h', \emptyset)$ instead of $U_m(h')$ for h' in $H - U$. In separating k in K from U, there are two possibilities. If $k(0) = h(0)$, then $U_1(k, \emptyset) \cap U$ is empty — again corresponding to the situation in $X(4)$. Otherwise, since $k \upharpoonright 1 \neq h \upharpoonright 1$, $U_1(k, \{h \upharpoonright 1\}) \cap U$ is empty. Therefore U is closed. Let $\mathcal{B}'$ consist of all singletons on I together with those members $U_n(x, t)$ of $\mathcal{B}$ where $\{x \upharpoonright 1\} \in t$. Then $\mathcal{B}'$ is also a base for X and is clopen.

Since X is T_1 and zero-dimensional, it is T_3 and furthermore Tychonoff.

(b): X has character 2^{ω_1} at each point of $H \cup K$ since that is the size of each S_n. Although X therefore is not developable and does not have a σ-disjoint or a σ-locally countable base, it is useful to reflect what countable structure X has. For each $n \geq 1$, let $\mathcal{G}_n^0 = \{U_n(h, \emptyset) \mid h \in H\}$, $\mathcal{G}_n^1 = \{U_n(k, \emptyset) \mid k \in K\}$,

$\mathcal{G}_n^2 = \{\{x\} \mid x \in I - \bigcup(\mathcal{G}_n^0 \cup \mathcal{G}_n^1)\}$, and $\mathcal{G}_n = \mathcal{G}_n^0 \cup \mathcal{G}_n^1 \cup \mathcal{G}_n^2$. Each collection $\mathcal{G}_n^e$ is pairwise disjoint for $e = 0, 1,$ or 2. As in $X(4)$, each $\mathcal{G}_n$ is locally countable, as $\mathcal{G}_{n+1}$ testifies. But in this example, $\bigcup_{n \geq 1} \mathcal{G}_n$ is not a base for X.

(c): Showing that X is para-Lindelöf resembles showing that $X(4)$ is para-Lindelöf. Let $\mathcal{V}$ be an open cover of X. For each f in F, let $c(f)$ be the smallest integer n such that both $U_n(f^\frown 0, t)$ and $U_n(f^\frown 1, t)$ are contained in members of $\mathcal{V}$ for some t in T_n. For any $n \geq 1$, let $A_n = \{f \upharpoonright n \mid c(f) = n\}$; for each ρ in A_n choose t_ρ in T_n so that both $U_n(f^\frown 0, t_\rho)$ and $U_n(f^\frown 1, t_\rho)$ refine members of $\mathcal{V}$ for some (every) f in $[\rho : F]$. For convenience, for each x in $H \cup K$, let $c(x) = c(x \upharpoonright \omega)$ and $t(x) = t_\rho$, where $\rho = x \upharpoonright c(x)$. Also, for every x in $H \cup K$, let $t'(x) = t(x) \cup \{A_i \mid 1 \leq i \leq c(x)\}$.

For each x in $H \cup K$, let $W(x)$ be the open set $U_{c(x)}(x, t'(x))$. Then $h \upharpoonright c(h)$ determines $W(h)$ for h in H; similarly, $k \upharpoonright c(k)$ determines $W(k)$ for k in K. Now let $\mathcal{W}^0 = \{W(h) \mid h \in H\}$, $\mathcal{W}^1 = \{W(k) \mid k \in K\}$, and $\mathcal{W}^2 = \{\{x\} \mid x \in I - \bigcup(\mathcal{W}^0 \cup \mathcal{W}^1)\}$. Let $\mathcal{W}$ be the open cover $\mathcal{W}^0 \cup \mathcal{W}^1 \cup \mathcal{W}^2$, refining $\mathcal{V}$.

The possibilities for intersection between members of $\mathcal{W}$ show a picture like that in $X(4)$. Both $\mathcal{W}^0 \cup \mathcal{W}^2$ and $\mathcal{W}^1 \cup \mathcal{W}^2$ are pairwise disjoint collections. Now suppose that $W(h)$ meets $W(k)$ where $h \in H$ and $k \in K$. To see that $c(h) = c(k)$, suppose this is false. Without loss of generality assume that $c(h) < c(k)$, and let $n = c(h)$. Choose (σ, τ, r) in $W(h) \cap W(k)$. A_n is in both $t'(h)$ and $t'(k)$. Thus $A_n \in r$ since $h \restriction n \in A_n$, but $A_n \notin r$ since $k \restriction n \notin A_n$. This is of course impossible, and so $c(h) = c(k)$.

Now we verify that $\mathfrak{w}$ is locally countable. For any x in I, $\{x\}$ meets at most two members of $\mathfrak{w}$. For every x in $H \cup K$, let $Z(x)$ be the open set $W(x) \cap U_{c(x)+1}(x, \emptyset)$. To check that each $Z(x)$ avoids all but countably many members of $\mathfrak{w}$, consider $Z(h)$ where $h \in H$ and $c(h) = n$. $Z(h)$ meets just one member of $\mathfrak{w}^0 \cup \mathfrak{w}^2$ — namely, $W(h)$. Because $Z(h)$ refines $W(h)$, every member of $\mathfrak{w}^1$ met by $Z(h)$ lies in $\mathfrak{w}_n^1$ — where $\mathfrak{w}_n^1 = \{W(k) \mid k \in K \text{ and } c(k) = n\}$ — and is thus contained in a member of $\mathcal{G}_n^1$. But since $Z(h)$ also refines $U_{n+1}(h, \emptyset)$, it meets only countably many members of $\mathcal{G}_n^1$, each of which contains at most one member of $\mathfrak{w}_n^1$. Thus $Z(h)$ meets only countably many members of $\mathfrak{w}^1$ and hence of $\mathfrak{w}$. So $\mathfrak{w}$ is locally countable, and therefore X is para-Lindelöf.

(d): To verify that X is not normal, let V_1 and V_2 be arbitrary open sets which contain the disjoint closed sets H and K, respectively. As in $X(4)$, we will see that V_1 and V_2 intersect and thus that H and K cannot be separated by disjoint open sets. With $\mathfrak{v}$ the open cover $\{V_1, V_2, I\}$, let $c(f)$ for each f in F, A_n for each $n \geq 1$, and t_ρ for each ρ in some A_n be as in the proof of (c). Let $A = \bigcup_{n \geq 1} A_n$. Since every f in

F has a restriction in A, by Lemma 3.2.11, choose $n \geq 1$ and a subset B of A_n so that B is n-full. Now, by Lemma 3.2.12, find a subset B' of B so that the family $(t_\rho : \rho \in B')$ is an n-full Δ-system with root family $(t_\theta : \theta \in \mathrm{pr}B')$. For each ρ in B', let $r_\rho = \bigcup_{i \leq n} \{E \in t_\rho \cap S_i \mid \rho \restriction i \in E\}$, the set of those members of t_ρ on which ρ votes yes. Define the "matching" function $M : B' \to \mathcal{P}(t_\emptyset)$ by letting $M(\rho) = r_\rho \cap t_\emptyset$, the set of those members of $t_\emptyset$ on which ρ votes yes, for each ρ in B'. Since there are only finitely many possibilities for $M(\rho)$, by Lemma 3.2.10, let C be an n-full subset of B' on which M is constant; thus every ρ in C votes the same way on the members of $t_\emptyset$. Finally, by Lemma 3.2.9, find two interlacing members σ and τ of C with $\sigma(0) \neq \tau(0)$. Let $r = r_\sigma \cup r_\tau$. In verifying that the point $(\sigma, \tau, r) \in V_1 \cap V_2$, consider E in t_σ. Suppose $E \in r_\tau$. Then $E \in t_\sigma \cap r_\tau$, a subset of $t_\sigma \cap t_\tau = t_\emptyset$. So $E \in r_\tau \cap t_\emptyset = r_\sigma \cap t_\emptyset$. Thus $E \in r$ iff $E \in r_\sigma$ iff σ votes yes on E. Similarly, given E in t_τ, $E \in r$ iff $E \in r_\tau$ iff τ votes yes on E. So, for any h in $[\sigma : H]$ and k in $[\tau : K]$, (σ, τ, r) belongs to $U_n(h, t_\sigma) \cap U_n(k, t_\tau)$ and in turn to $V_1 \cap V_2$. Therefore X is not normal.

(e): We now use the proof of (c) to show that X is meta-compact, as in Section 4.1. Given any open cover $\mathcal{V}$ of X, the locally countable open refinement $\mathcal{W}$ constructed in proving (c) is also point-finite. But more simply, let the function c on $F \cup H \cup K$, each A_n, each t_ρ, and the function t on $H \cup K$ be as in the proof of (c). Then let $\mathfrak{U}^0 = \{U_{c(h)}(h, t(h)) \mid h \in H\}$ and $\mathfrak{U}^1 = \{U_{c(k)}(k, t(k)) \mid k \in K\}$. Define $\mathfrak{U}^2$ and $\mathfrak{U}$ accordingly as in the proof that $X(4)$ is metacompact, and continue as in that proof. Notice that we could just as well have defined functions c and t independently on H and K.

(f): We also use the proof of (c) to show that X is countably paracompact, as in Section 4.1. For a countably infinite open cover $\mathcal{V} = \{V_i \mid i \geq 1\}$ of X, for each f in F let $c(f)$ be the smallest integer n such that both $U_n(f^\frown 0, t)$ and $U_n(f^\frown 1, t)$ are contained in sets V_i with $i \leq n$ for some t in T_n. Define each A_n; each t_ρ; and $c(x)$, $t(x)$, and $W(x)$ for each x in $H \cup K$ accordingly as in the proof of (c). Now continue as in the proof that $X(4)$ is countably paracompact. $\square$

In this example, the members of the sets T_n are essential in getting X to be para-Lindelöf and also countably paracompact. For each $i \geq 1$, identify P_i with the set G' of characteristic functions in Bing's example G by a one-one, onto function s_i; G' plays the same role here that T', the top level of the ω_1-Cantor tree T, plays in the example X of Chapter 4.

Let h in H and $n \geq 1$ be given. Any "nth level" open neighborhood of h — one whose intersection with $H \cup K$ is $[h \restriction n : H]$ and whose intersection with I is determined by its intersection with I_n — can be thinned down by an association with the open neighborhoods of $s_i(h \restriction i)$ in G for each $i \leq n$. More precisely, given t in T_n, suppose that $t \cap S_i = t_i$ for each $i \leq n$. For each $i \leq n$, the nth level open neighborhood $U_n(h, t)$ of h is associated with the open neighborhood $\bigcap\{U_{E,0} \mid E \in t_i$ and $h \restriction i \notin E\} \cap \bigcap\{U_{e,1} \mid E \in t_i$ and $h \restriction i \in E\}$ of $h \restriction i$ in G; we take the intersection of an empty collection to be the entire space G. Similarly, given k in K and $n \geq 1$, any nth level open neighborhood of k can be thinned down by an association with the open neighborhoods of $s_i(k \restriction i)$ in G for each $i \leq n$. If the neighborhood in G associated with $h \restriction i$ and $t \cap S_i$ is disjoint from that associated with $k \restriction i$ and $t' \cap S_i$ for some $i \leq n$, then $U_n(h, t)$ and $U_n(h, t')$ are disjoint.

Since the space G is normal, this thinning down provides enough separation to make X para-Lindelöf. For, as in Chapter 4, it provides a way of controlling the "levels" of intersecting open sets. On the other hand, G is not collectionwise Hausdorff since no uncountable subset of the discrete set G' can be separated by disjoint open sets. Therefore a collection of members t_ρ of T_n cannot distinguish between uncountably many functions ρ in any given P_n. So the thinning down does not provide enough separation to make X normal.

Section 5.2. A Collectionwise Nonnormal Topological T_4-Space.

Now we use the techniques of the previous section to build a topological space like the space Y of Section 4.2. Once again, this construction does not require any extra set-theoretic assumptions.

<u>5.2.1: Example.</u> For each $n \geq 1$, let $J_n = \{(\{\sigma, \tau\}, r)$ where $\{\sigma, \tau\} \subset P_n$ and $r \in T_n \mid$ (i) $\sigma(0) \neq \tau(0)$, and (ii) σ and τ interlace$\}$; let $J_n^* = \bigcup_{m \geq n} J_m$. Now let $J = J_1^*$ and $Y = F \cup J$.

Now we topologize Y. The points of J are isolated. For each $n \geq 1$, f in F, and t in T_n, let $U_n(f, t) = [f \restriction n : F] \cup \{(\{\sigma, \tau\}, r) \in J_n^* \mid$ (i) $\sigma \restriction n = f \restriction n$ or $\tau \restriction n = f \restriction n$, and (ii) given $E \in t \cap S_i$ for some $i \leq n$, $E \in r$ iff $f \restriction i \in E\}$; these sets are open. The set of all singletons on J together with these open sets make up the base $\mathcal{B}$ for Y.

The topological space Y has the following properties:

(a) Y is T_1 and zero-dimensional and also normal.

(b) Y has character 2^{ω_1}; therefore Y is not a Moore space and does not have a σ-disjoint or a σ-locally countable base.

(c) Y is para-Lindelöf.

(d) Y is not collectionwise normal.

(e) Y is metacompact.

(f) Y is countably paracompact.

In the discussion of this example, the topological space X referred to will be that of Example 5.1.4. Also, Y(4) will refer to the space of Example 4.2.1.

(a): Y is T_1 exactly as Y(4) is. Showing that Y is zero-dimensional is very similar to showing the same result for X. Consider $U = U_n(f,t)$ where $n \geq 1$, $f \in F$, $t \in T_n$, and $\{f \restriction 1\} \in t$; we verify that U is closed. Given y in J - U, $\{y\}$ separates y from U as in Y(4). In separating f' in F - U from U, there are two possibilities. If $f'(0) = f(0)$, separate f' from U as in Y(4) — using $U_m(f', \emptyset)$ instead of $U_m(f')$. Otherwise, since $f' \restriction 1 \neq f | 1$, $U_1(f', \{f \restriction 1\}) \cap U$ is empty. Let $\mathcal{B}'$ consist of all singletons on J together with those members $U_n(f,t)$ of $\mathcal{B}$ where $\{f \restriction 1\} \in t$. Then $\mathcal{B}'$ is a clopen base for Y.

To see that Y is normal, let A and B be disjoint closed subsets of Y. As in the analogous proof of Y(4), we assume without loss of generality that both A and B are contained in F. For each f in A, let c(f) be the smallest integer n such that $U_n(f,t)$ avoids B for some t in T_n, and let t(f) be such a t. Similarly, for each f in B, let c(f) be the smallest integer n such that $U_n(f,t)$ avoids A, and A for some t in T_n, and let t(f) be such a t. For each $n \geq 1$, let $A_n = \{f \restriction n | f \in A \text{ and } c(f) = n\}$ and $B_n = \{f \restriction n | f \in B \text{ and } c(f) = n\}$. For each f in $A \cup B$ with $c(f) = n$, let $t'(f) = t(f) \cup \{A_i \mid 1 \leq i \leq n\} \cup \{B_i \mid 1 \leq i \leq n\}$. Now let $U = \bigcup\{U_{c(f)}(f, t'(f)) \mid f \in A\}$ and $V = \bigcup\{U_{c(f)}(f, t'(f)) \mid f \in B\}$. Clearly U and V are open sets which contain A and B, respectively.

Certainly no point of F can be in $U \cap V$. To see that U and V are disjoint, suppose that $y = (\{\sigma, \tau\}, r) \in U \cap V$. Then $y \in U_{c(f)}(f, t'(f)) \cap U_{c(f')}(f', t'(f'))$ for some f in A and f' in B. Without loss of generality assume that $\sigma \restriction c(f) = f \restriction c(f)$, $\tau \restriction c(f') = f' \restriction c(f')$, and $c(f) \le c(f')$. Let $n = c(f)$. Then A_n belongs to both $t'(f)$ and $t'(f')$. So $A_n \in r$ since $f \restriction n \in A_n$, but $A_n \notin r$ since $f' \restriction n \notin A_n$. Since this is impossible, U and V are disjoint. Therefore Y is T_4.

(b): Y has character 2^{ω_1} at each point of F. Although Y therefore is not developable and does not have a σ-disjoint or a σ-locally countable base, we can reflect the countable structure that Y does have. Given any $n \ge 1$, let $\mathcal{G}_n^1 = \{U_n(f, \emptyset) | f \in F\}$, $\mathcal{G}_n^2 = \{\{y\} | y \in J - \bigcup \mathcal{G}_n^1\}$, and $\mathcal{G}_n = \mathcal{G}_n^1 \cup \mathcal{G}_n^2$. Once again, each $\mathcal{G}_n$ is locally countable, as $\mathcal{G}_{n+1}$ testifies. However, as for X, $\bigcup_{n \ge 1} \mathcal{G}_n$ is not a base for Y.

(c): We now modify the proof that X is para-Lindelöf to prove that Y is para-Lindelöf, as in Chapter 4. Given an open cover $\mathcal{U}$ of Y, for each f in F let $c(f)$ be the smallest integer n such that $U_n(f, t)$ is contained in a member of $\mathcal{U}$ for some t in T_n. Once again, for any $n \ge 1$, let $A_n = \{f \restriction n | f \in F \text{ and } c(f) = n\}$; for each ρ in A_n choose t_ρ in T_n so that $U_n(f, t_\rho)$ refines a member of $\mathcal{U}$ for any f in $[\rho : F]$. For convenience, for each f in F, let $t(f) = t_\rho$ where $\rho = f \restriction c(f)$; let $t'(f) = t(f) \cup \{A_i | 1 \le i \le c(f)\}$.

Now, for each f in F, let $W(f)$ be the open set $U_{c(f)}(f, t'(f))$, which is determined by $f \restriction c(f)$. Define $\mathcal{W}^1$, $\mathcal{W}^2$, and the open refinement $\mathcal{W}$ for $\mathcal{V}$ as in the analogous proof for $Y(4)$. As in the proof for X, $W(f)$ and $W(f')$ are disjoint if $c(f) \neq c(f')$ where f and f' are in F. As a result, $\mathcal{W}$ is locally countable; for each f in F, the open set $Z(f) = W(f) \cap U_{c(f)+1}(f, \emptyset)$ witnesses the local countability of $\mathcal{W}$ at f.

(d) - (f): The proof that X is not normal can be modified to prove that Y is not collectionwise normal, just as in Chapter 4; this is left to the reader. Similarly, the proofs showing that X is metacompact and countably paracompact can be modified to prove the same results for Y, just as in Chapter 4. □

Section 6.1. Properties Found in T_3, Para-Lindelöf, Nonparacompact Spaces.

By examining the examples of the previous two chapters and making some simple modifications, we can investigate which topological properties can go along with being T_3 and para-Lindelöf but not paracompact. These four main examples will be referred to, in order, as X(4), Y(4), X(5), and Y(5). Earlier notational definitions still hold, except where they are specifically overruled.

X(4) and Y(4), the examples using a normal ω_1-Cantor tree, depend on the existence of such a tree; for example, they can be constructed under $MA(\omega_1)$ but cannot under CH. However, in their favor, these spaces are first countable. In fact, each of them is developable and has a σ-locally countable base; only X(4) has a σ-disjoint base. On the other hand, X(5) and Y(5), the examples using Bing's G, are "real" examples. But these spaces fail to be first countable. In fact, they both have character 2^{ω_1}. The two nonnormal spaces, X(4) and X(5), are screenable; for each, the point-finite open refinement constructed in showing metacompactness is also σ-disjoint (in fact the union of two pairwise disjoint collections). The two normal spaces, Y(4) and Y(5), are not screenable; for each, the open cover used in showing that the space is not collectionwise normal has no σ-disjoint open refinement. While all four spaces are completely regular, none are collectionwise normal. All are metacompact and countably paracompact.

An elegantly simple technique devised by F. Burton Jones uses a space which is regular but not normal to construct a new space which is regular but not completely regular. By applying this technique to the spaces X(4) and X(5), we get para-Lindelöf T_3-spaces which are not completely regular.

<u>6.1.1: Construction</u> (Jones [7]). Let X be a topological space which is regular but not normal. Accordingly, let H and K be disjoint closed subsets of X which cannot be separated by disjoint open sets. Let N be the set of natural numbers $\{1, 2, \cdots\}$ with the discrete topology. Let Z' be the identification space $X \times N/\sim$ where the equivalence relation $\sim$ on $X \times N$ is generated by the rule $(x, n) \sim (x, n+1)$ if: $x \in H$ and n is even, or $x \in K$ and n is odd. Finally, let $Z = Z' \cup \{p\}$ where $p \notin Z'$. For each n in N, let $X_n = \{(x, n) | x \in X\}/\sim$, which is homeomorphic to X. To topologize Z, let $U \subset Z'$ be open in Z iff U is open in Z', or equivalently iff $U \cap X_n$ is open in X_n for each n in N. Also, if $U \subset Z$ with $p \in U$, let U be open iff $U \cap Z'$ is open and U contains $\bigcup_{i \geq n} X_i$ for some n in N.

While Z is regular, Jones shows that Z is not completely regular. For example, there can be no continuous function $f : Z \to [0,1]$ which is 0 at p and 1 at every point of the closed set X_1. This construction preserves many topological properties, including T_1-ness and para-Lindelöfness. □

<u>6.1.2: Example</u> ($MA(\omega_1)$). Let X be X(4), and use

X in Construction 6.1.1. Then Z is a para-Lindelof Moore space which is not completely regular. Like X, Z has a σ-disjoint, σ-locally countable base and is metacompact and countably paracompact. □

6.1.3: Example. Let X be X(5), and build Z by applying Construction 6.1.1 to X. Then Z is T_3 and para-Lindelöf but not completely regular. Like X, Z has character 2^{ω_1} and is metacompact and countably paracompact. □

We can also get T_3, para-Lindelöf spaces which are not metacompact by modifying, in turn, Y(4) and Y(5). We replace sets of two interlacing functions by sets of ω-many interlacing functions.

6.1.4: Example ($MA(\omega_1)$). For each $n \geq 1$, let $J_n = \{\{\sigma_i | i \in \omega\} \subset P_n | $ (i) $\sigma_i(0) \neq \sigma_j(0)$ when $i \neq j$, (ii) the functions σ_i are mutually interlacing, and (iii) the sequences $s_m(\sigma_i \restriction m) \restriction n$ for $1 \leq m \leq n$ are the same for every $i \in \omega\}$; let $J_n^* = \bigcup_{m \geq n} J_m$. Now let $J = J_1^*$ and $Y = F \cup J$.

Now we topologize Y. The points of J are isolated. For each $n \geq 1$ and f in F, the set $U_n(f) = [f \restriction n : F] \cup \{\{\sigma_i | i \in \omega\} \in J_n^* | \sigma_i \restriction n = f \restriction n$ for some $i \in \omega\}$ is open. The singletons on J together with these open sets provide a base for the topology on Y.

The space Y has all the properties mentioned for Y(4) except for metacompactness. Proving that Y is not metacompact is very similar to proving that Y(4) is not collectionwise normal. Here, again, we modify the argument showing that X(4) is not normal. For each $\alpha < \omega_1$, let $F_\alpha = \{f \in F \mid f(0) = \alpha\}$ and let U_α be an open set with $U_\alpha \cap F = F_\alpha$. I claim that the open cover $\mathfrak{U} = \{U_\alpha \mid \alpha < \omega_1\} \cup \{J\}$ has no point-finite open refinement. To see this, let $\mathfrak{V}$ be an arbitrary open refinement for $\mathfrak{U}$. With respect to the open cover $\mathfrak{V}$, define $c(f)$ for each f in F, A_n for each $n \geq 1$, and A as in the proof that $X(4)$ is not normal. Also let $n \geq 1$, $B \subset A_n$, $M : B \to ({}^n 2)^n$, and $C \subset B$ be as in that proof. Now, by Lemma 3.2.9, choose mutually interlacing members of C, σ_i for every $i \in \omega$, no two of which agree at 0. Let y be the point $\{\sigma_i \mid i \in \omega\}$. Then for each $i \in \omega$, $st(y, \mathfrak{V})$ meets $F_{\sigma_i(0)}$. But each member of $\mathfrak{V}$ meets at most one F_α. Since y thus lies in infinitely many members of $\mathfrak{V}$, $\mathfrak{V}$ is not point-finite. Therefore Y is not metacompact. □

<u>6.1.5: Example</u>. For each $n \geq 1$, let $J_n = \{(\{\sigma_i \mid i \in \omega\}, r)$ where $\{\sigma_i \mid i \in \omega\} \subset P_n$ and $r \in T_n \mid$ (i) $\sigma_i(0) \neq \sigma_j(0)$ when $i \neq j$, and (ii) the functions σ_i are mutually interlacing$\}$; let $J_n^* = \bigcup_{m \geq n} J_m$. Now let $J = J_1^*$ and $Y = F \cup J$.

Now we topologize Y. The points of J are isolated. For each $n \geq 1$, f in F, and t in T_n, the set $U_n(f,t) = [f \restriction n : F] \cup \{(\{\sigma_i \mid i \in \omega\}, r) \in J_n^* \mid$ (i) $\sigma_i \restriction n = f \restriction n$ for some $i \in \omega$, and (ii) given any $E \in t \cap S_i$ for some $i \leq n$, $E \in r$ iff $f \restriction i \in E\}$ is open. The singletons on J together with these open sets provide a base for the topology on Y.

The space Y has all of the properties mentioned for $Y(5)$ except for metacompactness. Showing that Y is not metacompact is analogous to showing the same result in Example 6.1.4. A proof, like the proof showing that $Y(5)$ is not collectionwise normal, is based on the argument showing that $X(5)$ is not normal. □

By taking disjoint unions of some of the para-Lindelöf T_3-spaces already discussed, we can get new ones which combine their negative properties. We have constructed para-Lindelöf T_3-spaces which are respectively not completely regular, completely regular but not normal, and normal but not collectionwise normal, and we have now eliminated metacompactness from the normal spaces. By taking a disjoint union with one of these modified spaces, we can eliminate metacompactness from any of the nonnormal spaces as well. In doing this, however, we also eliminate screenability; in fact, if a space is screenable and countably metacompact, then it is metacompact. Similarly, with a disjoint union, we can eliminate just screenability from any of the nonnormal spaces.

If we are willing to give up developability, we can (under $MA(\omega_1)$) eliminate σ-locally countable bases while retaining first countability. By a theorem of V. Fedorcuk [10], every paracompact T_2-space with a σ-locally countable base is metrizable. So any paracompact, nonmetrizable T_2-space, for example the Sorgenfrey line E, is T_3 and para-Lindelöf but has no σ-locally countable base (and hence no development, as every para-Lindelöf Moore space has a σ-locally countable base). Taking the disjoint union of either $X(4)$ or $Y(4)$ (under $MA(\omega_1)$) with E will leave a first countable, para-Lindelöf, nonparacompact T_3-space which has no σ-locally countable base and is not a Moore space.

By a theorem of E. Michael [8], a continuous, closed image of a paracompact T_2-space is itself paracompact. However, in the next example we see that a Moore space which is a continuous, closed image of a para-Lindelöf Moore space need not be para-Lindelöf.

<u>6.1.6: Example</u>. Let X be $X(5)$. Using the notation of Example 5.1.4, define an equivalence relation $\sim$ on X as follows. Define $\sim$ on I by letting $(\sigma, \tau, r) \sim (\sigma', \tau', r')$ iff $\sigma = \sigma', \tau = \tau'$, and $r \cap S_1 = r' \cap S_1$. Extend $\sim$ to an equivalence relation on X by letting $x \sim x$ for $x \in H \cup K$. Then the projection map $p : X \to X/\sim$ is a continuous, closed map. Although the image $X/\sim$ is a Moore space, like the space in [1], it is not para-Lindelöf. □

Section 6.2. Open Questions.

Below are some interesting open questions dealing with para-Lindelöf, nonparacompact T_3-spaces.

1. Without assuming any extra set-theoretic axioms, can one construct such a space which is first countable? □

2. Is there any such space which is not countably paracompact? □

3. Is there any such space which is collectionwise normal? (Diana Pike Palenz [9] has shown that every para-Lindelöf, monotonically normal space is paracompact. She also showed that every monotonically normal space with a σ-locally countable base is metrizable, an extension of Fedoruk's theorem.) □

4. Is there any such space which is normal as well as screenable? Is there any such space which is normal and has a σ-disjoint base? □

Glossary.

c.c.c.: A topological space has the c.c.c. (countable chain condition) if there is no collection of uncountably many mutually disjoint open subsets of the space.

collectionwise Hausdorff: A T_1-space X is collectionwise Hausdorff if whenever $\{x_\alpha \mid \alpha \in A\}$ is a discrete collection of points of X, there is a mutually disjoint collection $\{U_\alpha \mid \alpha \in A\}$ of open subsets of X with $x_\alpha \in U_\alpha$ for every $\alpha \in A$.

collectionwise normal: A topological space X is collectionwise normal if whenever $\{F_\alpha \mid \alpha \in A\}$ is a discrete collection of closed subsets of X, there is a mutually disjoint collection $\{U_\alpha \mid \alpha \in A\}$ of open subsets of X with $F_\alpha \subset U_\alpha$ for every $\alpha \in A$.

countably paracompact: A topological space is countably paracompact if every countable open cover of the space has a locally finite open refinement.

developable: A topological space is developable if it has a a development.

development: A development for a topological space X is a collection $\{\mathcal{G}_n \mid n \in \omega\}$ of open covers of X such that at each point $x \in X$, the collection $\{st(x, \mathcal{G}_n) \mid n \in \omega\}$ is a base for the topology of X at x.

discrete: A collection $\mathcal{F}$ of subsets of a topological space X is discrete if every point of X has a neighborhood which meets at most one member of $\mathfrak{F}$. A collection of points of X is discrete if the associated collection of singletons is discrete.

locally countable: A collection $\mathcal{U}$ of subsets of a topological space is locally countable if every point of the space has a neighborhood which meets only countably many members of $\mathcal{U}$.

locally finite: A collection $\mathcal{U}$ of subsets of a topological space is locally finite if every point of the space has a neighborhood which meets only finitely many members of $\mathcal{U}$.

metacompact: A topological space is metacompact if every open cover of the space has a point-finite open refinement.

meta-Lindelöf: A topological space is meta-Lindelöf if every open cover of the space has a point-countable open refinement.

monotonically normal: A T_1-space X is monotonically normal if there is a function H (known as a monotone normality operator) which assigns to each ordered pair (x, U), where $x \in X$ and U is an open neighborhood of x in X, an open set $H(x, U)$ such that for elements x and y of X and open sets U and V in X:

(i) if $x \in U$, then $x \in H(x,U)$;

(ii) if $x \in U \subset V$, then $H(x,U) \subset H(x,V)$; and

(iii) if $x \neq y$, then $H(x, X-\{y\}) \cap H(y, X-\{x\}) = \emptyset$.

Moore space: A T_3-space is a Moore space if it is developable.

paracompact: A topological space is paracompact if every open cover of the space has a locally finite open refinement.

para-Lindelöf: A topological space is _para-Lindelöf_ if every open cover of the space has a locally countable open refinement.

point-countable: A collection $\mathfrak{u}$ of subsets of a topological space is _point-countable_ if every point of the space belongs to only countably many member of $\mathfrak{u}$.

point-finite: A collection $\mathfrak{u}$ of subsets of a topological space is _point-finite_ if every point of the space belongs to only finitely many members of $\mathfrak{u}$.

σ-disjoint: A collection of subsets of a topological space is _σ-disjoint_ if it is the union of countably many collections, each of which is mutually disjoint.

σ-locally countable: A collection of subsets of a topological space is _σ-locally countable_ if it is the union of countably many locally countable collections.

σ-locally finite: A collection of subsets of a topological space is _σ-locally finite_ if it is the union of countably many locally finite collections.

σ-paracompact: A topological space is _σ-paracompact_ if every open cover of the space has a σ-locally finite open refinement.

σ-para-Lindelöf: A topological space is _σ-para-Lindelöf_ if every open cover of the space has a σ-locally countable refinement.

screenable: A topological space is _screenable_ if every open cover of the space has a σ-disjoint open refinement.

star: Given a collection $\mathcal{U}$ of subsets of a topological space X and a point $x \in X$, the <u>star</u> of x in $\mathcal{U}$, denoted by $st(x, \mathcal{U})$, is the union of all of those members of $\mathcal{U}$ to which x belongs.

star-countable: A collection $\mathcal{U}$ of subsets of a topological space is <u>star-countable</u> if each member of $\mathcal{U}$ has nonempty intersection with only countably many members of $\mathcal{U}$.

star-finite: A collection $\mathcal{U}$ of subsets of a topological space is <u>star-finite</u> if each member of $\mathcal{U}$ has nonempty intersection with only finitely many members of $\mathcal{U}$.

strongly collectionwise Hausdorff: A T_1-space X is <u>strongly collectionwise Hausdorff</u> if whenever $\{x_\alpha \mid \alpha \in A\}$ is a discrete collection of points of X, there is a discrete collection $\{U_\alpha \mid \alpha \in A\}$ of open subsets of X with $x_\alpha \in U_\alpha$ for every $\alpha \in A$.

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TITLE OF THESIS NONPARACOMPACTNESS IN PARA-LINDELOF SPACES

MAJOR PROFESSOR Mary Ellen Rudin

MAJOR DEPARTMENT Mathematics

MINOR(S) Computer Sciences

NAME Caryn Linda Navy

PLACE AND DATE OF BIRTH Brooklyn, New York; July 5, 1953

COLLEGES AND UNIVERSITIES: YEARS ATTENDED AND DEGREES

Massachusetts Institute of Technology, 1971-1975,Sb 1975

University of Wisconsin, Madison, 1975-1981,MA 1977,
Phd 1981

MEMBERSHIPS IN LEARNED OR HONORARY SOCIETIES

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