**Mathematical F ormula Handbook**

**1. Contents**



Bibliography; Physical Constants



Arithmetic and Geometric progressions; Convergence of series: the ratio test;

Convergence of series: the comparison test; Binomial expansion; Taylor and Maclaurin Series;

Power series with real variables; Integer series; Plane wave expansion



Scalar product; Equation of a line; Equation of a plane; Vector product; Scalar triple product;

Vector triple product; Non-orthogonal basis; Summation convention



Unit matrices; Products; Transpose matrices; Inverse matrices; Determinants; matrices;

Product rules; Orthogonal matrices; Solving sets of linear simultaneous equations; Hermitian matrices;

Eigenvalues and eigenvectors; Commutators; Hermitian algebra; Pauli spin matrices

1. Vector Calculus

Notation; Identities; Grad, Div, Curl and the Laplacian; Transformation of integrals



Complex numbers; De Moivre's theorem; Power series for complex variables.



Relations between sides and angles of any plane triangle;

Relations between sides and angles of any spherical triangle



Relations of the functions; Inverse functions







Standard forms; Standard substitutions; Integration by parts; Differentiation of an integral;

Dirac -'function'; Reduction formulae

1. Differential Equations

Diffusion (conduction) equation; Wave equation; Legendre's equation; Bessel's equation;

Laplace's equation; Spherical harmonics



1. Functions of Several Variables

Taylor series for two variables; Stationary points; Changing variables: the chain rule; Changing variables in surface and volume integrals - Jacobians

1. Fourier Series and Transforms

Fourier series; Fourier series for other ranges; Fourier series for odd and even functions;

Complex form of Fourier series; Discrete Fourier series; Fourier transforms; Convolution theorem;

Parseval's theorem; Fourier transforms in two dimensions; Fourier transforms in three dimensions





Finding the zeros of equations; Numerical integration of differential equations;

Central difference notation; Approximating to derivatives; Interpolation: Everett's formula;

Numerical evaluation of definite integrals



Range method; Combination of errors



Mean and Variance; Probability distributions; Weighted sums of random variables;

Statistics of a data sample Regression (least squares fitting)

**2. Introduction**

This Mathematical Formaulae handbook has been prepared in response to a request from the Physics Consultative Committee, with the hope that it will be useful to those studying physics. It is to some extent modelled on a similar document issued by the Department of Engineering, but obviously reflects the particular interests of physicists. There was discussion as to whether it should also include physical formulae such as Maxwell's equations, etc., but a decision was taken against this, partly on the grounds that the book would become unduly bulky, but mainly because, in its present form, clean copies can be made available to candidates in exams.

There has been wide consultation among the staff about the contents of this document, but inevitably some users will seek in vain for a formula they feel strongly should be included. Please send suggestions for amendments to the Secretary of the Teaching Committee, and they will be considered for incorporation in the next edition. The Secretary will also be grateful to be informed of any (equally inevitable) errors which are found.

This book was compiled by Dr John Shakeshaft and typeset originally by Fergus Gallagher, and currently by Dr Dave Green, using the typesetting package.

Version 1.5 December 2005.

**3. Bibliography**

Abramowitz, M. & Stegun, I.A., Handbook of Mathematical Functions, Dover, 1965.

Gradshteyn, I.S. & Ryzhik, I.M., Table of Integrals, Series and Products, Academic Press, 1980.

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Speigel, M.R., Mathematical Handbook of Formulas and Tables.

(Schaum's Outline Series, McGraw-Hill, 1968).

**4. Physical Constants**

Based on the "Review of Particle Properties", Barnett et al., 1996, Physics Review D, 54, p1, and "The Fundamental Physical Constants", Cohen & Taylor, 1997, Physics Today, BG7. (The figures in parentheses give the 1-standarddeviation uncertainties in the last digits.)

speed of light in a vacuum

c permeability of a vacuum permittivity of a vacuum elementary charge

Planck constant

Avogadro constant

unified atomic mass constant

mass of electron

mass of proton

Bohr magneton

molar gas constant

Boltzmann constant

Stefan-Boltzmann constant

gravitational constant

Other data

acceleration of free fall

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| --- | --- |
|  | (by definition) |
|  | (by definition) |
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(standard value at sea level)

**5. Series**

**6. Arithmetic and Geometric progressions**

A.P.   
G.P. for

(These results also hold for complex series.)

Convergence of series: the ratio test

**7. Convergence of series: the comparison test**

If each term in a series of positive terms is less than the corresponding term in a series known to be convergent, then the given series is also convergent.

**8. Binomial expansion**

If is a positive integer the series terminates and is valid for all : the term in is or where is the number of different ways in which an unordered sample of objects can be selected from a set of objects without replacement. When is not a positive integer, the series does not terminate: the infinite series is convergent for .

**9. Taylor and Maclaurin Series**

If is well-behaved in the vicinity of then it has a Taylor series,

where and the differential coefficients are evaluated at . A Maclaurin series is a Taylor series with

**10. Power series with real variables**

valid for all

valid for

valid for all values of

valid for all values of

valid for

valid for

valid for

**11. Integer series**

This last result is a special case of the more general formula,

Plane wave expansion

where are Legendre polynomials (see section 11 and are spherical Bessel functions, defined by

**12. Vector Algebra**

If are orthonormal vectors and then . [Orthonormal vectors orthogonal unit vectors.]

**13. Scalar product**

where is the angle between the vectors

Scalar multiplication is commutative: .

**14. Equation of a line**

A point lies on a line passing through a point and parallel to vector if

with a real number.

**15. Equation of a plane**

A point is on a plane if either

(a) , where is the normal from the origin to the plane, or

(b) where are the intercepts on the axes.

**16. Vector product**

, where is the angle between the vectors and is a unit vector normal to the plane containing and in the direction for which form a right-handed set of axes.

in determinant form

in matrix form

Vector multiplication is not commutative: .

**17. Scalar triple product**

**18. Vector triple product**

**19. Non-orthogonal basis**

Similarly for and .

**20. Summation convention**

implies summation over   
  
  
where

**21. Matrix Algebra**

**22. Unit matrices**

The unit matrix of order is a square matrix with all diagonal elements equal to one and all off-diagonal elements zero, i.e., . If is a square matrix of order , then . Also .

is sometimes written as if the order needs to be stated explicitly.

**23. Products**

If is a matrix and is a then the product is defined by

In general .

**24. Transpose matrices**

If is a matrix, then transpose matrix is such that .

**25. Inverse matrices**

If is a square matrix with non-zero determinant, then its inverse is such that .

where the cofactor of is times the determinant of the matrix with the -th row and -th column deleted.

**26. Determinants**

If is a square matrix then the determinant of is defined by

where the number of the suffixes is equal to the order of the matrix.

**27. matrices**

If then,

**28. Product rules**

(if individual inverses exist) (if individual matrices are square)

**29. Orthogonal matrices**

An orthogonal matrix is a square matrix whose columns form a set of orthonormal vectors. For any orthogonal matrix

**30. Solving sets of linear simultaneous equations**

If is square then has a unique solution if exists, i.e., if .

If is square then has a non-trivial solution if and only if .

An over-constrained set of equations is one in which has rows and columns, where (the number of equations) is greater than (the number of variables). The best solution (in the sense that it minimizes the error is the solution of the equations . If the columns of are orthonormal vectors then .

**31. Hermitian matrices**

The Hermitian conjugate of is , where is a matrix each of whose components is the complex conjugate of the corresponding components of . If then is called a Hermitian matrix.

**32. Eigenvalues and eigenvectors**

The eigenvalues and eigenvectors of an matrix are the solutions of the equation . The eigenvalues are the zeros of the polynomial of degree . If is Hermitian then the eigenvalues are real and the eigenvectors are mutually orthogonal. is called the characteristic equation of the matrix .

If is a symmetric matrix, is the diagonal matrix whose diagonal elements are the eigenvalues of , and is the matrix whose columns are the normalized eigenvectors of , then

If is an approximation to an eigenvector of then (Rayleigh's quotient) is an approximation to the corresponding eigenvalue.

**33. Commutators**

**34. Hermitian algebra**

Hermiticity  
Eigenvalues, real  
Orthogonality  
Completeness

$$

\text { Bra-ket form }

$$

Rayleigh-Ritz

Lowest eigenvalue

$$

\lambda\_{0} \leq \frac{\int \psi^{*} O \psi}{\int \psi^{*} \psi}

$$

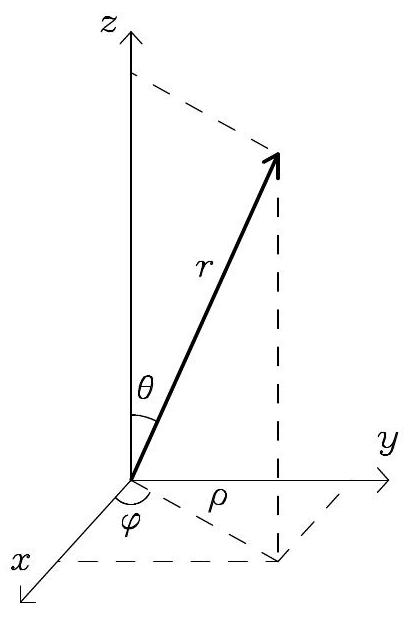
**35. Pauli spin matrices**

**36. Vector Calculus**

**37. Notation**

is a scalar function of a set of position coordinates. In Cartesian coordinates ; in cylindrical polar coordinates ; in spherical polar coordinates ; in cases with radial symmetry . is a vector function whose components are scalar functions of the position coordinates: in Cartesian coordinates , where are independent functions of .

In Cartesian coordinates ('del')



**38. Identities**

|  |  |  |  |
| --- | --- | --- | --- |
|  | Cartesian Coordinates | Cylindrical Coordinates | Spherical Coordinates |
|  |  |  |  |
| Vector |  |  |  |
| Gradient |  |  |  |
|  |  |  |  |
|  |  |  |  |
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**39. Transformation of integrals**

the distance along some curve ' ' in space and is measured from some fixed point.

a surface area

a volume contained by a specified surface

the unit tangent to at the point

the unit outward pointing normal

some vector function

the vector element of curve

the vector element of surface

Then

and when

Gauss's Theorem (Divergence Theorem)

When defines a closed region having a volume

also

When is closed and bounds the open surface ,

also

Green's Theorem

Green's Second Theorem

**40. Complex Variables**

**41. Complex numbers**

The complex number , where and is an arbitrary integer. The real quantity is the modulus of and the angle is the argument of . The complex conjugate of is

**42. De Moivre's theorem**

**43. Power series for complex variables.**

convergent for all finite convergent for all finite convergent for all finite principal value of

This last series converges both on and within the circle except at the point .

This last series converges both on and within the circle except at the points .

This last series converges both on and within the circle except at the point .

**44. Trigonometric Formulae**

**45. Relations between sides and angles of any plane triangle**

In a plane triangle with angles , and and sides opposite , and respectively,

diameter of circumscribed circle.

area where

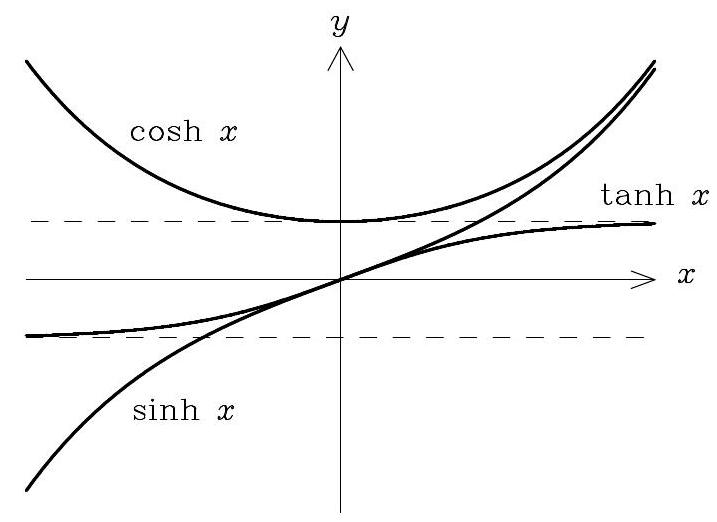
**46. Relations between sides and angles of any spherical triangle**

In a spherical triangle with angles , and and sides opposite , and respectively,

**47. Hyperbolic Functions**

valid for all

valid for all



For large positive :

For large negative :

**48. Relations of the functions**

**49. Inverse functions**

for   
  
for   
  
for   
  
for   
  
for   
  
for

**50. Limits**

as if any fixed

as any fixed

as as

If then (l'Hôpital's rule)

**51. Differentiation**

**52. Integration**

**53. Standard forms**

for

for and even

for and odd

if

if

**54. Standard substitutions**

If the integrand is a function of: substitute:

If the integrand is a rational function of or or both, substitute and use the results:

If the integrand is of the form: substitute:

**55. Integration by parts**

**56. Differentiation of an integral**

If is a function of containing a parameter and the limits of integration and are functions of then

Special case,

**57. Dirac -'function'**

If is an arbitrary function of then .

if , also

**58. Reduction formulae**

**59. Factorials**

Stirling's formula for large .

For any , etc.

For any .

Trigonometrical

If are integers,

and can therefore be reduced eventually to one of the following integrals

Other

If then .

**60. Differential Equations**

**61. Diffusion (conduction) equation**

Wave equation

**62. Legendre's equation**

solutions of which are Legendre polynomials , where , Rodrigues' formula so etc.

**63. Recursion relation**

Orthogonality

**64. Bessel's equation**

solutions of which are Bessel functions of order .

Series form of Bessel functions of the first kind

The same general form holds for non-integer .

**65. Laplace's equation**

If expressed in two-dimensional polar coordinates (see section 4), a solution is

where are constants and is a real integer.

If expressed in three-dimensional polar coordinates (see section 4) a solution is

where and are integers with are constants;

is the associated Legendre polynomial.

If expressed in cylindrical polar coordinates (see section 4), a solution is

where and are integers; are constants.

**66. Spherical harmonics**

The normalized solutions of the equation

are called spherical harmonics, and have values given by

Orthogonality

**67. Calculus of Variations**

The condition for to have a stationary value is , where . This is the Euler-Lagrange equation.

**68. Functions of Several Variables**

If then implies differentiation with respect to keeping constant.

where are independent variables. is also written as or when the variables kept constant need to be stated explicitly.

If is a well-behaved function then etc.

If ,

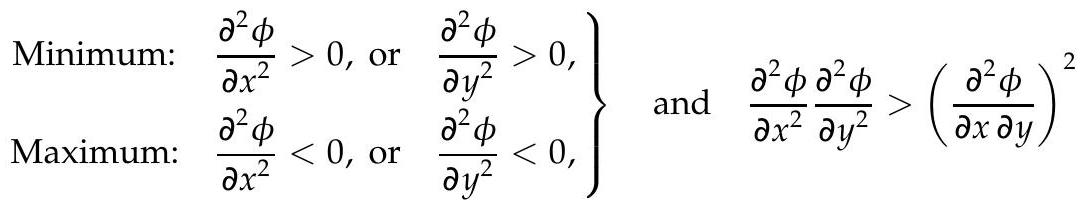
**69. Taylor series for two variables**

If is well-behaved in the vicinity of then it has a Taylor series

where and the differential coefficients are evaluated at

**70. Stationary points**

A function has a stationary point when . Unless , the following conditions determine whether it is a minimum, a maximum or a saddle point.



Saddle point:

If the character of the turning point is determined by the next higher derivative.

Changing variables: the chain rule

If and the variables are functions of independent variables then

**71. Changing variables in surface and volume integrals - Jacobians**

If an area in the plane maps into an area in the plane then

The Jacobian is also written as . The corresponding formula for volume integrals is

**72. Fourier Series and Transforms**

**73. Fourier series**

If is a function defined in the range then

where the coefficients are

$$

(m=1, ···, M)

$$

with convergence to as for all points where is continuous.

**74. Fourier series for other ranges**

Variable , range , (i.e., a periodic function of time with period , frequency ).

where

Variable , range ,

where

**75. Fourier series for odd and even functions**

If is an odd (anti-symmetric) function [i.e., ] defined in the range , then only sines are required in the Fourier series and . If, in addition, is symmetric about , then the coefficients are given by (for even), (for odd). If is an even (symmetric) function [i.e., ] defined in the range , then only constant and cosine terms are required in the Fourier series and . If, in addition, is anti-symmetric about , then and the coefficients are given by (for even), (for odd)

[These results also apply to Fourier series with more general ranges provided appropriate changes are made to the limits of integration.]

**76. Complex form of Fourier series**

If is a function defined in the range then

with taking all integer values in the range . This approximation converges to as under the same conditions as the real form.

For other ranges the formulae are:

Variable , range , frequency

Variable , range

**77. Discrete Fourier series**

If is a function defined in the range which is sampled in the equally spaced points , then

where the coefficients are

$$

(m=1, ···, N-1)

$$

each summation being over the sampling points .

**78. Fourier transforms**

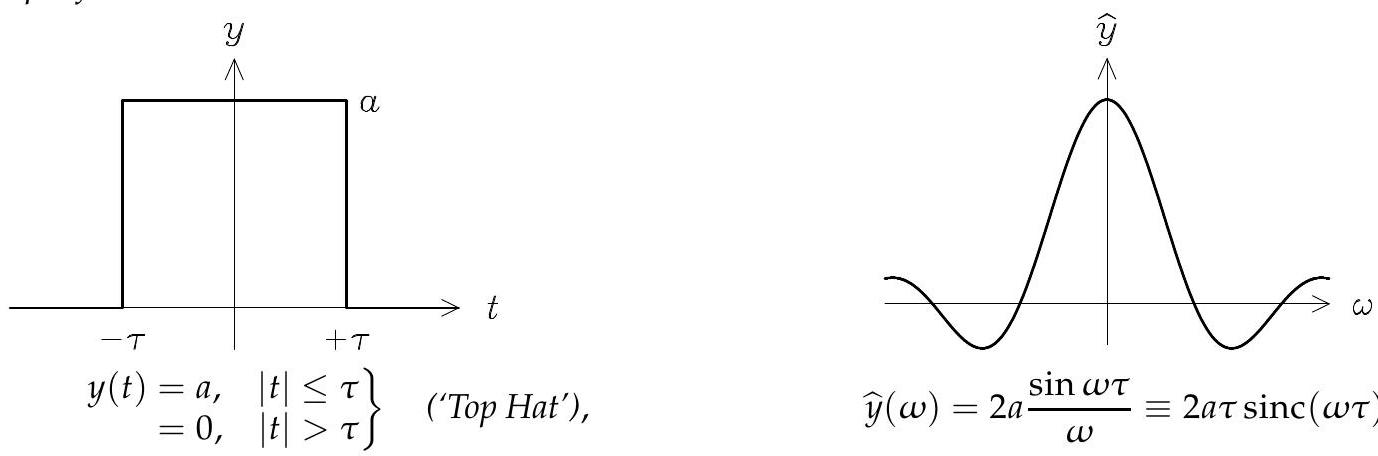
If is a function defined in the range then the Fourier transform is defined by the equations

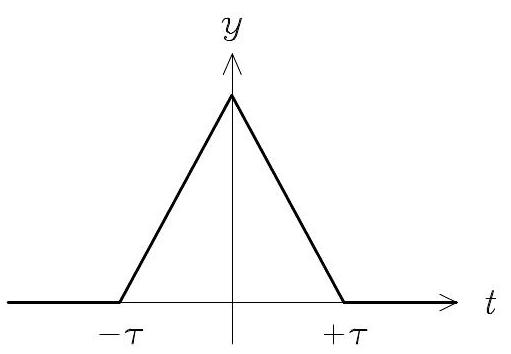
If is replaced by , where is the frequency, this relationship becomes

If is symmetric about then

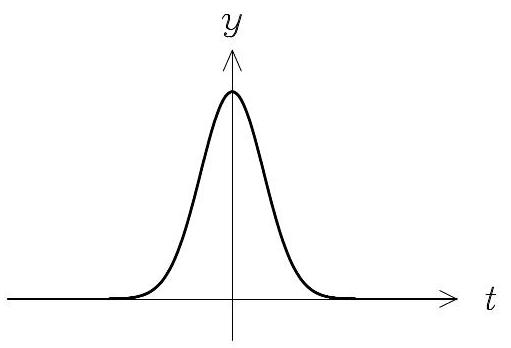
If is anti-symmetric about then

Specific cases

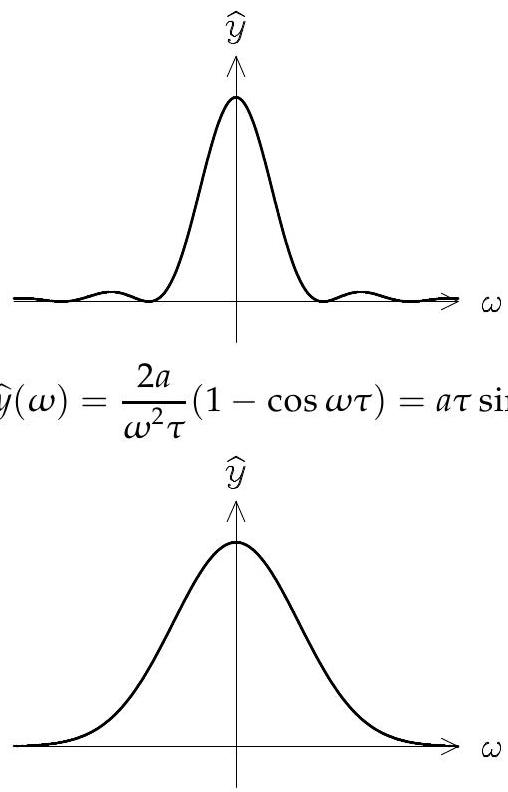




('Saw-tooth'),



(Gaussian)



where

**79. Convolution theorem**

If then .

Conversely, .

Parseval's theorem

(if is normalised as on page 21)

**80. Fourier transforms in two dimensions**

Fourier transforms in three dimensions

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|  | 1 |
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Examples

**81. Laplace Transforms**

If is a function defined for , the Laplace transform is defined by the equation

[Note that if for then the Fourier transform of is .]

**82. Numerical Analysis**

**83. Finding the zeros of equations**

If the equation is and is an approximation to the root then either

(Newton)

are, in general, better approximations.

**84. Numerical integration of differential equations**

If then

(Euler method)

Putting

(improved Euler method)

then

**85. Central difference notation**

If is tabulated at equal intervals of , where is the interval, then and

**86. Approximating to derivatives**

**87. Interpolation: Everett's formula**

where is the fraction of the interval between the sampling points and . The first two terms represent linear interpolation.

**88. Numerical evaluation of definite integrals**

**89. Trapezoidal rule**

The interval of integration is divided into equal sub-intervals, each of width ; then

Simpson's rule

The interval of integration is divided into an even number (say ) of equal sub-intervals, each of width ; then

These have the general form

For (exact for any cubic).

For (exact for any quintic).

**90. Treatment of Random Errors**

Sample mean

Residual:

Standard deviation of sample:

Standard deviation of distribution:

Standard deviation of mean:

Result of measurements is quoted as .

**91. Range method**

A quick but crude method of estimating is to find the range of a set of readings, i.e., the difference between the largest and smallest values, then

This is usually adequate for less than about 12 .

**92. Combination of errors**

If (with , etc. independent) then

So if

(i)   
  
(ii) or ,  
  
(iii) ,  
  
(iv) ,  
  
(v) ,

**93. Statistics**

**94. Mean and Variance**

A random variable has a distribution over some subset of the real numbers. When the distribution of is discrete, the probability that is . When the distribution is continuous, the probability that lies in an interval is , where is the probability density function.

**95. Probability distributions**

Error function:

Binomial: where .

Poisson: , and

Normal:

**96. Weighted sums of random variables**

If then . If and are independent then .

Statistics of a data sample

$$

\text { Sample variance } s^{2}=\frac{1}{n} \sum\left(x\_{i}-\bar{x}\right)^{2}=\left(\frac{1}{n} \sum x\_{i}{2}\right)-\bar{x}{2}=E\left(x{2}\right)-[E(x)]{2}

$$

**97. Regression (least squares fitting)**

To fit a straight line by least squares to pairs of points , model the observations by , where the are independent samples of a random variable with zero mean and variance .

Sample statistics: .

Estimators: at residual variance),

where residual variance .

Estimates for the variances of and are and .

Correlation coefficient: